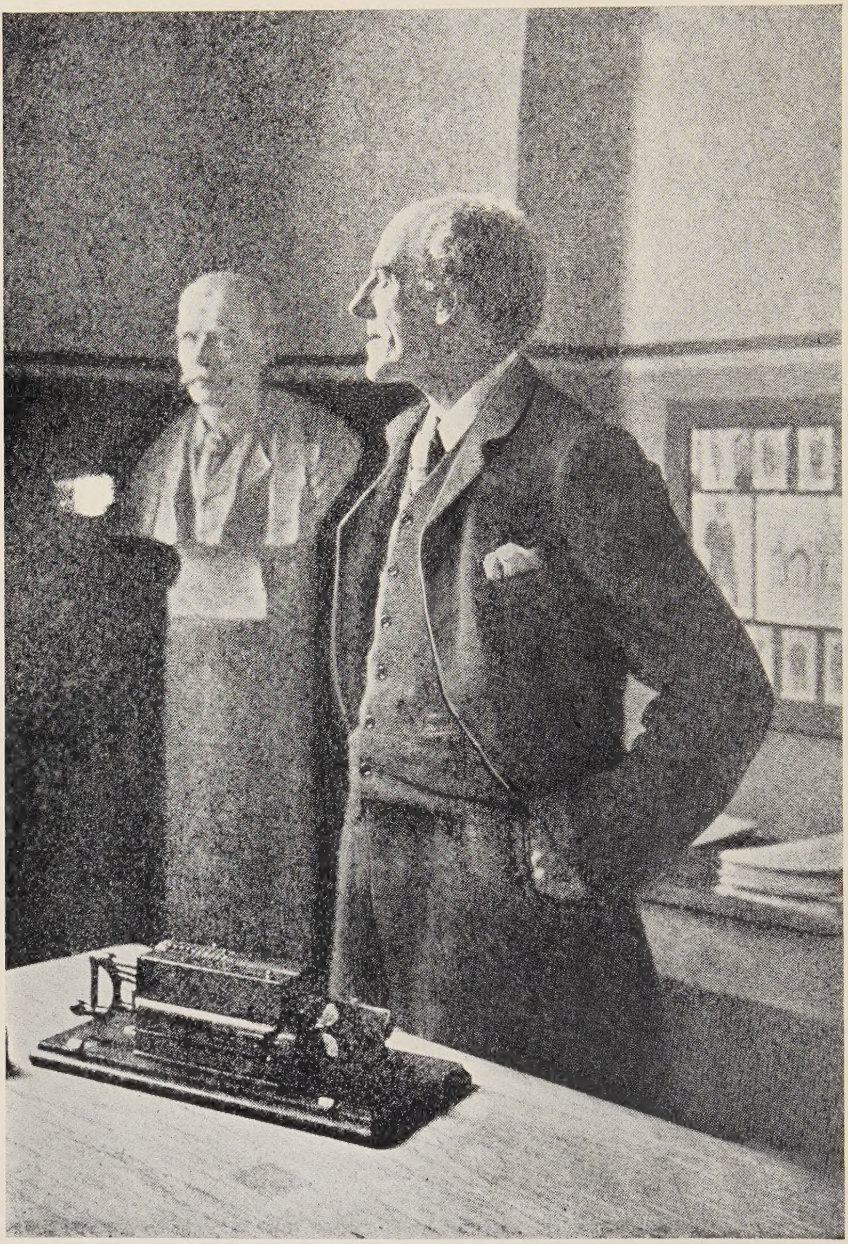
STUDIES IN THE HISTORY OF
STATISTICAL METHOD



Professor Karl Pearson
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STUDIES IN THE HISTORY OF STATISTICAL METHOD

WITH SPECIAL REFERENCE TO CERTAIN EDUCATIONAL PROBLEMS

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"The time may not be very remote when it will be understood that for complete initiation as an efficient citizen of one of the new great complex world wide states that are now developing, it is as necessary to be able to compute, to think in averages and maxima and minima, as it is now to be able to read and to write."

H. G. WELLS, *Mankind in the Making.*

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CHAPTER I

Introduction

Educational statistics is the offspring of a varied ancestry. The greed of ancient kings enumerating their people for taxation; the panic of an English sovereign during the London plague; the cupidity of professional gamblers; the scientific ardor of the psycho-physicists; the labors of mathematicians and astronomers and physicists and actuaries; the enthusiasm of the students of social phenomena; the disciplined imagination of the biologists; and the vision of educators planning a new science of education; from these has educational statistics descended.

The rapidity with which statistics is becoming the language of educational research makes it appear highly desirable that there should be available to students of statistics some account of the historical development of topics closely related to educational statistical theory and practice. At present there is but little accessible material except the original documents from which a student may learn by whom any particular term or formula or technique was first used, and what its subsequent history has been. There is a prevailing idea that much of statistical theory is of more recent origin than is actually the case. Formulas published a hundred years ago are spoken of as though they were the discoveries of this century, if not of this decade. The same formulas are often discovered independently by different men who are each unaware of the work of the other. Unnecessary and redundant terms have been coined, and terms have been used in a sense never intended by their originators, largely because the field is wide, some of the sources are difficult to obtain, and not many people have been able to take the time to ascertain exactly what their predecessors have written.

This study attempts to present the modern use of statistics against a background of the work of DeMoivre, Bernoulli, Gauss, Laplace, Quetelet, Galton, Ebbinghaus, Fechner, and many others. It is hoped that such a survey may be of use to students in various ways. It will furnish a historical perspective which will enable them better to understand and evaluate the present statistical practices and techniques employed in professional education, and it will provide a list of sources helpful to any student desiring to do intensive reading on a particular topic. It may also be of service to the man about to publish a new formula or a new term, by helping him discover what terms and formulas have already been proposed.

This is not a textbook in statistical theory, no attempt being made to develop any topic in such a way that it would be entirely clear to one who had no previous familiarity with it. Such readers are referred to the numerous texts in statistical theory and method now available. The use of mathematical symbols and formulas has been avoided wherever feasible, but it is manifestly impossible to discuss the history of statistical theory in language that is entirely non-technical.

In any study of so extended a topic, it is inevitable that a brief statement must fail to cover the entire field, and one of the most tormenting problems confronting the writer of history is always the decision as to what can be omitted, and what emphasis should be placed on the different portions of the mass of material contesting for attention. The reader interested in any special phase of statistics is almost certain to feel that his particular interest has been slighted. The consideration of certain important topics has, for the present, been regretfully relinquished. Among these are the following: (1) The effect of statistical methods upon modern educational practice; (2) The history of psycho-physics and its influence upon the use of statistics in education; (3) The history of scale making, including Fechner's law and the method of just perceptible differences; (4) The symbolism of statistical theory; (5) The use of statistics prior to 1900 in studies of school problems; (6) Computational devices, short cuts, and mechanical aids; (7) The application of the normal curve in educational studies; (8) A full treatment of methods of measuring correlation other than the Pearson product-moment method; (9) Curve fitting; (10) Index numbers and (11) Graphic methods. Any one of these might be made the subject of extended inquiry, and without some such study no picture of the development of statistical method can be considered complete.

It was originally planned to present a relatively full treatment of the introduction of statistical methods into the field of education, and with that purpose in view the writer collected a large amount of critical and biographical data concerning the work of men prominent in statistical research in America at the present time. As the study progressed, however, the difficulty of making a fair appraisal of current work became more and more apparent, and it was finally decided to omit any extended discussion of recent material, placing the emphasis rather upon origins. This omission is not to be construed as an indication that the writer is lacking in appreciation of the contributions of men now living, but rather as a recognition of the general principle that a historical study is seldom able to furnish a true picture of the present situation. It has seemed desirable, therefore, to dis-

cuss in considerable detail works which are not generally accessible, and to speak with relative brevity of recent books and papers.

The generous assistance of many persons has made the task of gathering the data a pleasant one. During the course of the work, the writer has had conferences with Dr. James McKeen Cattell, Professor Robert Chaddock, Mr. Arne Fisher, Dr. Charles C. Grove, Professor Truman L. Kelley, Dr. Willford I. King, Mr. Edwin W. Kopf, Professor William A. McCall, Professor H. L. Rietz, Professor Harold Rugg, Dr. Beardsley Ruml, Professor George D. Strayer, Professor Percival M. Symonds, Professor E. L. Thorndike, and Professor R. S. Woodworth. Others gave their time to reply in writing to questions asked them, and the writer has been greatly helped by letters from Dr. Leonard P. Ayres, Dr. Frank P. Bachman, Dr. B. R. Buckingham, Professor Charles L. Cooley, Professor Stuart A. Courtis, Dr. John Cummings, Dr. Charles B. Davenport, Dean Eugene Davenport, Professor Walter C. Eells, Dr. Roland P. Falkner, Professor Irving Fisher, Professor James W. Glover, Professor Karl J. Holzinger, Professor Truman Kelley, Professor Walter S. Monroe, Dr. Arthur S. Otis, Professor Karl Pearson, Professor Giles M. Ruch, Dean Carl E. Seashore, Professor Horace Secrist, Professor Carl Spearman, Professor Lewis M. Terman, Professor Godfrey Thomson, Professor L. L. Thurstone, Professor Herbert A. Toops, Professor E. B. Wilson, and Professor Clifford Woody. Mr. Henry Rinsland, Dr. Vera Sanford and Dr. Ella Woodyard read and criticized portions of the manuscript. Professor William D. Reeve read the entire manuscript and made valuable suggestions concerning the general organization of the study. The writer is permanently indebted to Professor Henry Alford Ruger and to Professor David Eugene Smith for the original stimulus to undertake the work, for help and criticism during its progress, and for standards of scholarship which are at once a challenge and a despair.

CHAPTER II

The Normal Curve

1. ORIGIN IN THE THEORY OF PROBABILITY

Early Traces. The origin of the normal curve of distribution lies much farther back in the past than most students of education realize. Not long ago the writer had an opportunity to test the extent of the popular misconceptions on this subject, by asking a class of about one hundred graduate students in a course in educational statistics to tell in which twenty-five year period they thought the normal curve was discovered. Six or eight placed the date in the twentieth century; the majority—perhaps thinking of Galton—ascribed it to the latter half of the preceding century; while only three or four wished to hazard an estimate as early as the first quarter of the nineteenth century. The correct date is November 12, 1733.

Before considering the normal curve itself, it is necessary to speak briefly of the background from which it sprang, namely the mathematical theory of probability. In this, as in most of the topics in educational statistics, other men have labored and we are now entering into the fruits of the labors of astronomers, mathematicians, physicists, biologists, psychologists, anthropologists, economists, statesmen, actuaries, and professional gamblers. The treatment of the theory of probability given here is intended as an outline of certain features in its development which may furnish students of statistics, and in particular of educational statistics, with a background that will assist them to understand and evaluate the ensuing discussion of the normal curve.[1]

[1] For a brief account of the history of probability see D. E. Smith, *History of Mathematics*, II, 528–530 (New York, 1925), hereafter referred to as Smith, *History*. More extended accounts may be found in the following: I. Todhunter, *A History of the Mathematical Theory of Probability, from the Time of Pascal to that of Laplace*, 624 p.(Cambridge, 1865), hereafter referred to as Todhunter, *History*; C. Gouraud, *Histoire du Calcul des Probabilités depuis ses origines jusqu'a nos jours*, 148 p., (Paris, 1848), hereafter referred to as Gouraud, *Histoire*; E. Czuber, "Die Entwicklung der Wahrscheinlichkeitstheorie und ihrer Anwendungen," *Jahresbericht der Deutscher Mathematiker—Vereinigung*, VII, 279 p. (Leipzig, 1899), hereafter referred to as Czuber, *Entwicklung*; the articles on "Probability" in the 7th, 9th, and 11th editions of the *Encyclopaedia Britannica;* the article on "Probability" in the *Library of Useful Knowledge*, presumably written by Lubbock and Bethune, but published anonymously; the articles by De Morgan on "Theory of Probabilities," in the *Encyclopaedia Metropolitana*, 1838, in *Mathematical Papers of De Morgan*, 393–490. See also Florian Cajori, *A History of Mathematics*, London and New York, 2nd ed., 1926.

Traces in the Orient. Before 1600 there are altogether not more than six or eight recorded problems which show even a faint intimation of the idea of a theory of probability. The Chinese writer Yüan Yüan[2] criticised a still older writer, Sun-Tze[3] for this curious reason: "Many unnecessary details appear in his works on mathematics, such as a certain absurd problem, which surely cannot be attributed to him, on the probability that an expected child will turn out to be a boy or to be a girl." Although this problem appeared absurd to Yüan Yüan, it did not so impress Arbuthnot, Nicolas Bernoulli, Poisson, and other more recent writers, who considered it quite seriously.

The Gambling Problem. The first reference to the theory of probability in a European work is a statement in a commentary (Venice, 1477) on Dante's *Divine Comedy* concerning the different throws which can be made with three dice.[4] The first writer to introduce a problem on gambling into a mathematical work was Luca Pacioli. In his *Sūma* (1494), he gave the first version[5] of the problem concerning the equitable division of the stakes between two players of unequal skill when the game is interrupted before its conclusion. This problem was repeated and amplified in all the works on probability for two hundred years. Cardan (1501–1576), who was himself an experienced gambler, wrote a sort of gambler's handbook called *De Ludo Aleae*, published in 1663 in the first volume of his collected works.[6]

In view of the very conspicuous rôle which astronomers played two centuries later in developing the mathematical theory of probability, it is interesting to observe that both Kepler and Galileo made brief reference to the subject of chance.[7]

None of the foregoing offered any general treatment or enunciated any principles; indeed most of them gave incorrect solutions for the very simple and isolated problems which they considered.

First Scientific Work on Probability. The real history of the theory of probability begins in the 17th century. Its foundations were well laid in

[2] P. L. Van Hée, "The Ch'ou=Jen Chuan of Yüan Yüan, *Isis*, VIII (1926), 106.

[3] "Père Vanhée puts the date as probably the 1st century A.D., while Biernatzki (p. 21) says that Sun-Tzī may have lived 220 B.C." Smith, *History*, I, 141.

[4] Smith, *History*, II, 529; Todhunter, *History*, p. 1; Libri, *Histoire des sciences mathématiques en Italie*, II, 188.

[5] For the original words, see Smith, *History*, II, 529, footnote.

[6] Todhunter, *History*, pp. 1–4.

[7] Todhunter, *History*, pp. 4–6.

1654 by two of the most distinguished mathematicians then living, Blaise Pascal[8] and Pierre de Fermat,[9] although neither of them made public anything on the subject. Vivid contrasts have marked the course of development of the theory, but none more striking than the incident which was the immediate cause for its discovery.[10] The Chevalier de Méré, a gambler whom Leibniz described as having unusual ability "even for the mathematics" appears to have proposed certain questions to Pascal, who was even then on the point of becoming a religious recluse. Among these questions was the celebrated "Problem of Points" concerning the division of stakes between two players who separate without completing their game. Pascal and Fermat exchanged numerous letters on this subject during the year 1654, and in the course of this correspondence they generalized the problem more and more, until at its close, that which had first appeared as a mere source of perplexity to a gambler had been elevated to a mathematical concept of great import.[11]

It is with something of surprise that we observe that this remarkable correspondence attracted but little immediate attention. The interest of Europe was soon diverted toward the work of Newton and Leibniz, so that the theory of probability was temporarily overlooked by the more prominent mathematicians.

Three years after this exchange of letters, there appeared a small tract by Christiaan Huygens,[12] entitled *De Ratiociniis in Ludo Aleae*,[13] which gave

[8] Blaise Pascal (1623–1662), after winning immortal renown as a mathematician and a physicist, retired at the age of twenty-five to a life of religious contemplation.

[9] Fermat (1608–1665) is best known for his work on the theory of numbers. He did not write for publication, but communicated his discoveries in letters to his friends, Pascal, Descartes, Mersenne, Roberval, and others.

[10] "Un problême relatif aux jeux de hasard, proposé à un austère janséniste par un homme du monde a été l'origine du calcul des probabilités." Poisson, *Recherches sur la Probabilités des Jugement*, p. 1.

[11] The portion of this correspondence which has been preserved, has been quoted in some detail by Todhunter, pp. 7–21. Three of the original letters from Pascal to Fermat, dated July 29, August 24, and October 27, 1654, may be found in *Varia Opera Mathematica D. Petri de Fermat*, pp. 179–188 (Toulouse, 1679). These same letters and also three from Fermat to Pascal, one undated, and the others of dates August 29 and September 25, and a letter on the same subject from Fermat to Carcavi, August 29, 1654, are to be found in *Œuvres de Blaise Pascal*, pp. 371–431 (Paris, 1908). Translations of several of these letters are to be found in the *Source Book in Mathematics* (New York, 1929). The first letter from Pascal to Fermat proposing to him the question asked by the Chevalier de Meré has been lost.

[12] Christiaan Huygens (1629–1695) was a famous Dutch physicist and mathematician.

[13] This was printed in Van Schooten's *Exercitationum mathematicarum libri quinque* (Leyden, 1657). Todhunter says that Van Schooten, who had been Huygens' teacher, translated the manuscript from the Dutch in which it was first written.

a mathematical treatment of the chances of winning certain card and dice games. This tract is important because it was the first printed work on the subject of games of chance, and also because it served to stimulate Jacques Bernoulli, Montmort, and De Moivre.

Within the next few years several problems relative to the outcome of games of chance were proposed and solved in the journals of various learned societies by Jacques Bernoulli, Montmort, De Moivre, Arbuthnot, Francis Roberts and others. A work of disputed authorship,[14] entitled *Of the Laws of Chance* (London, 1692), contained a translation of Huygens' treatise, a good many problems concerning games of chance, and an attempt at "a Calculation of the Quantity of Probability founded on Experience," such as the probability that an expected child will be a boy, or the probability that a given person will die within the year.

Bernoulli and the *Ars Conjectandi*. One of the great names closely associated with the theory of probability is that of the Swiss mathematician, Jacques Bernoulli,[15] celebrated author of *Ars Conjectandi*. Born at the close of the year in which Pascal and Fermat were writing the famous letters in which they established the fundamental principles of probability, Bernoulli became the author of the first book devoted wholly to that subject. The *Ars Conjectandi*[16] was published posthumously in 1713, being edited

[14] Montucla, Lubbock and Bethune, and Galloway ascribe it to one Benjamin Motte. Todhunter is of the opinion that the author was Arbuthnot.

[15] Jacques Bernoulli (or James Bernoulli, as the English usually write the name) (1654–1705) was the first of the nine mathematicians in the famous Bernoulli family. All of them won a measure of distinction, while Jacques, his brother Jean (or *John*), and his nephews Daniel and Nicolas, achieved world-wide renown.

[16] *Ars Conjectandi, Opus Posthumum Accedit Tractatus de Seriebus Infinitis et Epistola Gallice Scripta de Ludo Pilae Reticularis*, Basel (1713).

The following translations or commentaries are available:

L. G. F. Vastel, *L'Art de Conjecturer, traduit du Latin de Jacques Bernoulli, avec les Observations éclairissements et additions*, Caen (1801). (This is a translation of the first part only.)

R. Haussner, *Wahrscheinlichkeitsrechnung (Ars Conjectandi) von Jakob Bernoulli*, Ostwald's Klassiker der Exacten Wissenschaften, pp. 107–108 (Leipzig, 1899). (This is a translation of all four parts.)

Maseres, *The Doctrine of Permutations and Combinations, being an Essential and Fundamental Part of the Doctrine of Chances* (1795). (This is a translation of the second part only.)

Todhunter, *History*, pp. 56–77.

Gouraud, *Histoire*, pp. 23–29.

Pearson, "James Bernoulli's Theorem," *Biometrika* XVII (1925), 201–210. (This is a critical analysis of one particular aspect of the work of Bernoulli, namely his work on the use of the binomial expansion in calculating probabilities.)

by Nicolas Bernoulli who himself made numerous contributions to the theory of probability. The editor says in his two page preface that he does not feel competent to complete or revise the work and that it is in substantially the form in which it was left by his uncle.

In many ways this book is a landmark in the history of probability. The fact that a mathematician who had attained a considerable reputation for his work in several branches of pure mathematics wrote an entire book on probability is testimony to the growing importance and respectability of the subject. It contained a very definite suggestion of the idea of inverse probability, so essential to modern statistical theory and practice. It emphatically maintained that there is a great field for the application of the theory of probability to civil, moral, and economic affairs. Had the author been able to complete this section of the book, regrettably left unfinished at his death, the world might not have had to wait until the 19th century for a Quetelet. On the whole, however, it may be just as well that the theory remained the property of astronomers and mathematicians until it had been established on a scientific foundation, before its utility and manifold applications to human interests were disclosed. Bernoulli also endeavored to base the theory of probability upon definitions, a praiseworthy attempt but somewhat unconvincing in its outcome. Most important of all, he attempted in *Pars Quarta* to prove that if the number of observations is made sufficiently large, any previously determined degree of accuracy can be achieved.

It appears that Nicolas Bernoulli made an attempt to induce other mathematicians to work on his uncle's suggestion of applying the theory of probability in new realms. At the end of the preface to the first edition of *The Doctrine of Chances* (1718), De Moivre remarked that some years after his own *De Mensura Sortis* had been printed, there came out the posthumous work of Bernoulli, "wherein the Author has shewn a great deal of Skill and Judgment, and perfectly answered the Character and great Reputation he hath so justly obtained. I wish I were capable of carrying on a Project he had begun, of applying the Doctrine of Chances to Oeconomical and Political Uses, to which I have been invited, together with Mr. Montmort, by Mr. Nicolas Bernoully: I heartily thank that Gentleman for the good Opinion he has of me; but I willingly resign my share of that Task into better Hands, wishing that either he himself would prosecute that Design, he having formerly published some successful Essays of that Kind, or that his Uncle, Mr. John Bernoully, Brother to the Deceased, could be prevailed upon to bestow some of his Thoughts upon it, he being known to be perfectly well qualified in all Respects for such an Undertaking."

JACQUES BERNOULLI (1654–1705)
From the collection of Professor David Eugene Smith

Ars Conjectandi has been extravagantly praised by both Gouraud[17] and Todhunter. It is surprising that the latter, who had apparently perused both Bernoulli's *Pars Quarta* and the translation of De Moivre's *Approximatio* which was included in the second and third editions of *The Doctrine of Chances*,[18] could ascribe so much merit to Bernoulli's method of inequalities, and could ignore so completely the graceful, scholarly, and highly original solution of De Moivre.

Ars Conjectandi is in four parts. *Pars Prima* reproduces the treatise by Huygens on games of chance, with comments and solutions by Bernoulli. *Pars Secunda,* a treatise on permutations and combinations, begins by saying that the endless complexity seen both in the works of Nature and in the affairs of men, which is the final cause for all the remarkable beauty of the Universe, had its source in the varying groupings and arrangements of single parts, and that the theory of combinations and permutations is the working cause by which exist the wisdom of the philosopher, the accuracy of the historian, the diagnosis of the physician, and the cleverness of the politician, since all of these rest upon conjecture. Here is also found his definition of "Permutations,"[19] a term invented by him. The Bernoulli numbers,[20] useful in so many forms of numerical integration, are to be found on page 97 of this section. On the same page is his formula for the sum of the powers of the first r positive numbers, also helpful to the statistician who utilizes higher mathematical analysis. In the same section (p. 87) there is given a table resembling Pascal's triangle, but which the author calls "Tabula combinationum, seu numerorum figuratorum. Exponentes combinationum." He says that the properties of this table of figurate numbers had not, so far as he was aware, been pointed out by earlier writers. This statement, surprising in view of the fact that Pascal's work on the Arithmetic Triangle appeared in 1665, and that Leibniz mentioned it in a letter to Bernoulli in April 1705, is explained by Haussner as due to the fact that Bernoulli was sick during most of the latter part of his life. *Pars Tertia* deals with the applications of the theory of combinations to different plays in games of chance. *Pars Quarta,* giving Bernoulli's idea of the application of the preceding theory to civil, economic, and moral

[17] *Histoire,* pp. 28–29.

[18] See page 14.

[19] "Permutationes rerum voco Variationes, juxta quas fervatâ eâdem rerum multitudine ordo situsque inter ipsas diversimode permutatur," p. 74.

[20] Mr. Jekuthial Ginsburg has compiled an extensive bibliography on these numbers, which has not yet been published. His translation of Bernoulli's work on these numbers appears in the *Source Book in Mathematics,* edited by Professor David Eugene Smith (New York, 1929).

affairs, is the section of great historical importance, and also, unfortunately, is the section left unfinished at the death of the author. Bernoulli says that probability is a degree of certainty and differs from it as the part from the whole.[21] He uses the term *morally certain* to apply to a case where the probability is .99 or perhaps .999, and suggests that it would be well if bounds could be set to moral certainty[22] for the convenience of judges. But it is Bernoulli's use of the binomial expansion in *Pars Quarta* with which we are chiefly concerned. He was not trying to find an equation for the ordinate of the curve to which that expansion approximates as the exponent is indefinitely increased, and so he did not get the equation of the normal curve. Nor was he trying to find an approximate expression for the sum of a given number of terms of that expansion, and so he did not get anything that corresponds to the probability integral, so indispensable to modern statisticians. He wished to take a given number of terms—say n—on either side of the maximum term, and to have the ratio of the sum of these $2n$ terms to the sum of the remaining terms in the two tails of the expansion larger than some arbitrarily determined value, say c. Apparently he ignored the maximum term altogether, but its inclusion would merely increase the size of the ratio. Clearly, if n is made large enough, the probability that this ratio will exceed c may be made to approach certainty, no matter what value has been assigned to c. The goal of Bernoulli's solution is to find a limiting value which n must exceed in order that the ratio shall exceed c, and he attains this minimum limit of n by an elaborate use of inequalities such as is found in no modern text on probability. An account of the involved and somewhat awkward reasoning by which he reached this result may be read in Pearson's article on "James Bernoulli's Theorem."[23] Nothing which we might say could add anything to Pearson's analysis, and as it is easily accessible, we shall not repeat here. Pearson also contrasts Bernoulli's solution with the modern approach to the same problem, and points out the shortcomings of the former, which he says "would ruin either an insurance society or its clients if it were adopted."

[21] Voltaire (1772) wrote: "J'ignore pourquoi l'auteur de l'article *Probabilité* dans le grand Dictionnaire encyclopédique admet une demi-certitude. Il me semble qu'il n'y a pas plus de demi-certitude que de demi-vérité. Une chose est vraie ou fausse, point de milieu. Vous êtes certain ou incertain. L'incertitude étant presque toujours le partage de l'homme, vous vous détermineriez très-rarement, si vous attendiez une démonstration." *Essai sur les probabilités en fait de justice*, p. 3. The philosophical question, raised here has been discussed by many writers. See Keynes's *Treatise on Probability* (London, 1921).

[22] Compare the use of *practical certainty* when a difference is as great as three times its standard deviation, by some modern writers.

[23] *Biometrika* XVII (1925), 201–210.

Without doubt Bernoulli had gone a long way in advance of anything else which had been written on the subject of probability up to that time, and rightly felt that for novelty and usefulness, as well as difficulty, this section exceeded all other portions of his book.[24] It is regrettable that Todhunter should have claimed too much for Bernoulli, so that now it seems necessary to discount his contributions because they are less fundamental than Todhunter's account implies. For example, on page 57, Todhunter, referring to a letter from Bernoulli to Leibniz, says: "James Bernoulli then proceeds to speak of the celebrated theorem which is now called by his name." Reference to the original[25] reveals a very interesting correspondence that foreshadows Bayes's Theorem, but nothing that could well be interpreted as Bernoulli's Theorem, unless it is the statement which translated reads: "It remains to be discovered how by increasing the number of observations we increase the probability, so that at length, a certain probability being given, we should find the ratio between the cases differing from the true ratio less than that probability. In truth the problem has, so to speak, asymptotes." (pp. 77, 78). Todhunter's other references to "Bernoulli's Theorem" are equally difficult to justify when the sources are consulted. He states that on the second page of Bernoulli's *Lettre à un Amy*[26] is "a very distinct statement of the use of the celebrated theorem known by the name of Bernoulli." Reading the letter, we are unable to tell whether Todhunter thought that theorem related to inverse probability or to the increase in the accuracy of predictions with an increase in the number of observations, and in any case there is no "distinct statement."

Montmort. After the death of Bernoulli, but before the posthumous publication of *Ars Conjectandi*, appeared Montmort's[27] *Essai d'Analyse sur les Jeux de Hazards*,[28] a work intended more for mathematicians than for professional gamesters.

[24] "Hoc igitur ist illud Problema, quod evulgandum hoc loco proposui, postquam jam per vicennium pressi, cujus tum novitas, tum summa utilitas cum pari conjuncta difficultate omnibus reliquis hujus doctrinae capitibus pondus pretium superaddere potest. Ejus autem solutionem priusquam tradam, paucis objectiones diluam quas Viri quidam docti contra haec placita moverunt," p. 227.

[25] A letter from Bernoulli to Leibniz, written from Basel, Oct. 3, 1703. See Leibniz, *Opera Omnia.*

[26] *Lettre à un Amy, sur les Parties du Jeu de Paume*, bound with *Ars Conjectandi*, at the end of the volume.

[27] Pierre-Rémond de Montmort (1678–1719).

[28] First edition 1708, second edition 1714.

Early Work of De Moivre. Between the first and second editions of Montmort's essay on games of chance and before the appearance of *Ars Conjectandi*, De Moivre[29] published "De Mensura Sortis."[30] This memoir he expanded later into *The Doctrine of Chances: or, A Method of Calculating the Probabilities of Events in Play*.[31] The first edition of this was dedicated to Newton, who was then president of the Royal Society. In the preface to the first edition, De Moivre says that he undertook to write "De Mensura Sortis" about seven years earlier because of the desire and encouragement of Mr. Francis Robartes,[32] who proposed to him some problems more difficult than any in *L'Analyse des Jeux de Hazard*, and who then asked him to organize a general method and to lay down rules for their solution. At that time De Moivre had read only Huygens' treatise and "a little English Piece done by a very ingenious Gentleman" whom he failed to name. Later he read Montmort's work cursorily, but did not appear to regard it highly.

The Doctrine of Chances is in reality a gambler's manual, giving a systematic presentation of the arithmetic principles upon which are based the solution of problems concerning the advantage of players and the size of wager which may be laid in a wide variety of games of chance. There is some treatment of infinite series, and it is interesting to discover here our

[29] Abraham De Moivre (1667–1754) was obliged to leave France after the revocation of the Edict of Nantes, and at the age of eighteen took refuge in London, where he spent the remainder of his life. Here he made a simple living by private tutoring in mathematics and by solving mathematical puzzles and problems for wealthy patrons. Quite naturally among the questions for whose answer men of wealth were willing to pay were problems concerning their chances at cards or dice or roulette, and this may account for the very extensive knowledge of the rules of these games which his work on probability shows. Besides his contributions to the theory of probability, De Moivre is chiefly known for his work on trigonometry and on annuities. See Smith, *History*, I, 450.

[30] "De Mensura Sortis, seu, de Probabilitate Eventum in Ludis a Casu Fortuito Pendentibus," *Philosophical Transactions*, XXVII (1711), 213–264.

[31] First edition, 1718, quarto, xiv + 175. Second edition, enlarged, 1738, large quarto, xiv + 258. Third edition, further enlarged, 1756, large quarto, xii + 348. Italian Edition, *La Dottrina degli Azzardi applicata ai Problemi Della Probabilita della Vita, Della Pensioni, Vitalizie, Reversioni, Tontine, ec. Di Abramo Moivre* (Milan, 1776). This edition contains no material relating to games of chance, but only those sections of the third edition which deal with annuities, life insurance, mortality tables and the like. Thus it contains no material from the first edition of *The Doctrine of Chances*. Much new material has been added by the translators. In the new material is a bibliography of works on life insurance and annuities.

[32] Presumably the Honorable Francis Roberts, who wrote "An Arithmetical Paradox, concerning the Chances of Lotteries," *Philosophical Transactions*, XVII (1693), 677–681.

Abraham De Moivre (1667–1754)
From *Biometrika*, Vol. XVII, by permission of Professor Karl Pearson

rule for expressing the general term of a numerical function by the use of differences:[33]

$$u_n = u_0 + n\Delta u_0 + \frac{n(n-1)}{2!}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 u_0 + \dots$$

De Moivre is considering the case "if there be any Series of Terms whose last Differences are = 0. Let the number denoting the rank of the difference be *n;* then the Index of the Relation of each Term to as many of the preceding ones as there are Units in *n*, will be expressed by the Coefficients of the Binomial *a-b* raised to the Power *n*, omitting the first." A bit of practical advice offered in the preface to this first edition appears worthy of repetition: "On this Occasion, I must take notice to such of my Readers as are well vers'd in Vulgar Arithmetik, that it would not be difficult for them to make themselves Masters, not only of all the Practical Rules in this Book, but also of more useful Discoveries, if they would take the small Pains of being acquainted with the bare Notation of Algebra, which might be done in the hundredth part of the Time that is spent in learning to read Shorthand."

2. THE DISCOVERY OF THE NORMAL CURVE

De Moivre's Approximatio. The intensive study which De Moivre gave to this work, together with his applications of the binomial theorem, led him a few years later—probably about 1721—to discover a formula for the ratio between the middle term and the sum of all the terms of $(1 + 1)^n$, and thus to become the discoverer of the normal curve. Many of the recent treatises on probability and sampling approach the matter by a method quite similar to De Moivre's use of the binomial expansion.

This formula, first published November 12, 1733, is the *fons et origo* of the normal curve. In 1730, De Moivre had brought out his *Miscellanea Analytica* and three years later he presented privately to a few friends a brief paper of seven pages with the title *Approximatio ad Summam Terminorum Binomii* $\overline{a+b}^n$ *in Seriem expansi.* The discovery of this exceedingly rare document is due to Pearson,[34] who writes, "Many copies of this work (*Miscellanea*) have attached to them a Supplementum with separate pagination, ending in a table of 14 figure logarithms of factorials from 10! to 900!

[33] See Whittaker and Robinson, *The Calculus of Observations* (London, 1924), formula (1), p. 11.

[34] See "Historical Note on the Normal Curve of Errors," *Biometrika*, XVI (1924), 402–404.

by differences of 10. But only a *very few* copies have a second supplement, also with separate pagination (pp. 1–7) and dated Nov. 12, 1733. This second supplement could only be added to copies sold three years after the issue of the original book, and this accounts for its rarity."[35]

In this obscure treatise on abstract mathematics, written in Latin nearly two centuries ago, and supposed by its author to have no practical implications outside the realm of games of chance, in this brief supplement now so rare that only two copies have been reported extant, we have the first formulation of the momentous concept of a law of errors. We like to imagine De Moivre sitting at a table with a small group of cronies in some London coffee house, imparting to them the discovery which he had made through his knowledge of the properties of infinite series and his interest in the fortunes of the gambler. And we like to imagine the astonished incredulity with which they would have greeted a prophecy that there would come a time when that theorem would powerfully affect the thinking of the world on all its social problems, when it would enter the schools and shape the policies of educators, and when its aid would be invoked in thousands of investigations in sciences whose very names were then unknown.

Translation of the *Approximatio*. In 1738 De Moivre printed his own translation of the *Approximatio* in the second edition of *The Doctrine of Chances*, introducing it with the remark, "I shall here translate a Paper of mine which was printed November 12, 1733, and communicated to some Friends, but never yet made public, reserving to myself the right of enlarging my own Thoughts, as occasion shall require." The importance of this

[35] In a footnote, Pearson adds the fact that "All three copies of this work in the University College Library have the first, only one the second supplement. The sole copy in the British Museum Library has the first but not the second. Of the three copies at Cambridge (University Library) one has the first supplement only, the other two neither supplement. . . . The Royal Society has two copies, one has the first, the other neither supplement. The Bodleian Library has a single copy with neither supplement. The University Library, Edinburgh, has two copies, one has the first, the other neither supplement." Archibald (*infra*) says: "Another copy of Moivre's original pamphlet is in the Preussische Staatsbibliothek, Berlin." So far as the writer knows, no other copies have been found. Ginsburg has made a careful search in the New York Public Library but has not found it there in any of De Moivre's works. Through the courtesy of the Library of the University College, University of London, I have obtained a photostat copy of the *Approximatio*. Now, since the publication of Archibald's "A Rare Pamphlet of Moivre and some of his Discoveries," *Isis*, VIII (Oct. 1926), which contains a similar reproduction of these rare pages, this is generally accessible.

Nov. 12. 1733.

APPROXIMATIO AD SUMMAM TERMINORUM BINOMII $\overline{a+b}^n$ in Seriem expansi,

Autore *A. D. M.* R. S. S.

QUanquam solutio Problematum ad sortem spectantium non raro exigit ut plures Termini Binomii $\overline{a+b}^n$ in summam colligantur; attamen in potestatibus excelsis res adeo laboriosa videtur, ut perpauci hoc opus aggredi curaverint; *Jacobus* & *Nicolaus Bernoulli* viri Doctissimi primi quod sciam tentarunt quid sua industria in hoc genere præstare posset, in quo etiamsi uterque propositum summa cum laude sit assecutus, aliquid tamen ultra potest requiri, hoc est approximatio ad summam; non enim tam de approximatione videntur fuisse solliciti quam de assignandis certis limitibus quos Summa Terminorum necessario transcenderet. Quam vero viam illi tenuerint, breviter in Miscellaneis meis exposui * quæ consulat Lector si vacat, quod ipsi tamen scripserint melius erit fortasse consulere: Ego quoque in hanc disquisitionem incubui; quod autem eo me primum impulit non profectum fuit ab opinione me cæteros anteiturum, sed ab obsequio in Dignissimum virum qui mihi autor fuerat ut hæc susciperem; Quicquid est, novas cogitationes prioribus subnecto, sed eo ut connexio postremorum cum primis melius appareat, mihi necesse est ut pauca jampridem a me tradita denuo proferam.

I. Duodecim jam sunt anni & amplius cum illud inveneram; si Binomium $1+1$ ad potestatem n permagnam attollatur, ratio quam Terminus Medius habet ad summam Terminorum omnium, hoc est ad 2^n, ad hunc modum poterit exprimi $\frac{2A \times \overline{n-1}^n}{n^n \sqrt{n-1}}$, ubi A eum numerum exponit cujus Logarithmus

A

* Vide Miscellanea Analytica pag. 96. 97. 98. 99.

TITLE PAGE OF DE MOIVRE'S *Approximatio*
By courtesy of the Library of University College, University of London

document, combined with its inaccessibility, appears to justify rather extensive quotation from it.[36]

"Although the solution of Problems of Chance often require that several Terms of the Binomial $(a + b)^n$ be added together, nevertheless in very high Powers the thing appears so laborious, and of so great difficulty that few people have undertaken the Task; for besides *James* and *Nicolas Bernoulli*, two great Mathematicians, I know of no body that has attempted it; in which, though they have shewn very great skill, and have the praise which is due to their Industry, yet some things were farther required; for what they have done is not so much an Approximation as the determining of very wide limits, within which they demonstrated that the Sum of the Terms was contained. Now the Method which they followed has been briefly described in my *Miscellanea Analytica* which the reader may consult if he pleases, unless they rather chuse, which perhaps would be the best, to consult what they themselves have writ upon that Subject: for my part, what made me apply myself to the Inquiry was not out of opinion that I should excell others, [in which however I might have been forgiven;] but what I did was in compliance to the desire of a very worthy Gentleman and good Mathematician, who encouraged me to it; I now add some thoughts to the former; but in order to make their connexion the clearer, it is necessary for me to resume some few things that have been delivered by me a pretty while ago.

"I. It is now a dozen years or more since I had found what follows; If the Binomial $1 + 1$ be raised to a very high Power denoted by n, the ratio which the Middle Term has to the Sum of all the Terms, that is to 2^n, may be expressed by the Fraction $\frac{2A\ (n-1)^n}{n^n\sqrt{n-1}}$, wherein A represents the number of which the Hyperbolic Logarithm is $\frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$, &c. but because the Quantity $(n-1)^n$ or $\left(1-\frac{1}{n}\right)^n$ is very nearly given when n is a high Power, which is not difficult to prove, it follows that in an infinite Power, that Quantity will be absolutely given, and represent the number of which the Hyperbolic Logarithm is -1; . . . . and now suppose that

[36] These quotations are taken from De Moivre's translation of 1738, but they have been carefully checked against the original Latin version, and wherever the English introduces any new material that fact is indicated by inclosing the same in square brackets. When the particular phrasing of De Moivre does not seem to be of special interest, it has been paraphrased for the sake of brevity.

B represents the Number of which the Hyperbolic Logarithm is $1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680}$ &c., that Expression will be changed into $\frac{2}{B\sqrt{n}}$. When I first began that inquiry, I contented Myself to determine at large the Value of B, which was done by the addition of some Terms of the above written Series; but as I perceived that it converged but slowly, and seeing at the same time that what I had done answered my purpose tolerably well, I desisted from proceeding further, till my worthy Friend *Mr. James Stirling*, who had applied himself after me to that inquiry, found that the Quantity B denote the Square-root of the Circumference of a Circle whose Radius is Unity, so that if the Circumference be called c, the Ratio of the Middle Term to the Sum of all the Terms will be expressed by $\frac{2}{\sqrt{nc}}$."

Here De Moivre has stated two highly important facts. The first is that not Stirling but De Moivre was the discoverer of "Stirling's approximation for the value of factorial x,"[37] which is employed in deriving the equation for the normal curve from the binomial expansion. The second important fact stated here is that De Moivre knew that the maximum ordinate of the curve of error is $y = \frac{1}{\sigma\sqrt{2\pi}}$, for nc is equivalent to $8\sigma^2\pi$ under the circumstances of De Moivre's problem.

The Formula for the Curve of Error. Next De Moivre announces the formula for the curve of error, $y = y_o\, e^{-\frac{x^2}{2\sigma^2}}$, in these words: "This being admitted,[38] I conclude that if m or $\frac{1}{2}n$ be a Quantity infinitely great, then the Logarithm of the Ratio, which a Term distant from the middle by the Interval l has to the Middle Term, is $\frac{-2ll}{n}$."

The proportion of the total area included between the positive and negative values of that number which we now term the standard deviation, De Moivre found to be 0.682688. As he had no tables of the probability

[37] Stirling announced the formula in his *Methodus Differentialis*, p. 137 (London, 1730). It may also be found in the second edition (1764), p. 137, and in an English edition by H. Holliday, p. 121, (London, 1749).

[38] He has just stated that the logarithm of the ratio of the middle term to a term l units distant from it is approximately

$$(m + l - \tfrac{1}{2}) \log (m + l - 1) + (m - l + \tfrac{1}{2}) \log (m - l + 1) - 2\, m \log m + \log \frac{m+1}{m}.$$

integral, but was obliged to rely wholly on the method of mechanical quadratures to compute the area under the curve, this is a very good approximation, the error being no greater than one point in the sixth place. He gave no name to the number $\frac{1}{2}\sqrt{n}$, which is equivalent to the standard deviation when $p = q = \frac{1}{2}$, but in the translation of 1738 he said that $\sqrt{n}$ "will be as it were the Modulus by which we are to regulate our Estimation." In this version he also made use of the idea—though not the name—of the probable error, and said that the limits such that between their positive and negative values half of the errors may be expected to fall, are located at a distance of approximately $\frac{1}{4}\sqrt{2n}$ on either side of the middle term. Thus his value for the probable error was approximately 0.707σ instead of 0.67449σ. He also found the probability of an error numerically less than 3σ to be 0.99874, which exceeds the correct value by only about one-seventh of 1 per cent.

Probability and Theology. In the version of 1738, and to a still greater extent in the posthumously published edition of 1756, De Moivre connected his theory of probability with theology, urging that this tendency of events to conform to law argues a Great First Cause. "And thus in all cases it will be found, that although Chance produces irregularities, still the Odds will be infinitely great, that in process of Time, those Irregularities will bear no proportion to the recurrency of that Order which naturally results from Original Design. . . . . Again, as it is thus demonstrable that there are, in the constitution of things, certain Laws according to which Events happen, it is no less evident from Observation, that these Laws serve to wise, useful and beneficient purposes, to preserve the stedfast Order of the Universe, to propagate the several Species of Beings, and furnish to the sentient Kind such degrees of happiness as are suited to their State. . . . . Yet there are Writers, of a Class indeed very different from that of *James Bernoulli*, who insinuate as if the *Doctrine of Probabilities* could have no place in any serious Enquiry; and that studies of this kind, trivial and easy as they be, rather disqualify a man for reasoning on every other subject. Let the Reader chuse."

In these later editions, De Moivre displays an eagerness both to free the subject of probability from the opprobrium attaching to it because of its intimate connection with the art of gambling, and also to establish the theological doctrine of a divine order working through human affairs and exhibiting itself in the regularity of statistical ratios.[39]

[39] Cf. Süssmilch, *Die Göttliche Ordnung in den Veränderungen des menschlichen Geschlechts* (1742).

Comment by Pearson. Pearson observes[40] that Todhunter "missed entirely the epochmaking character of the 'Approximatio' as well as its enlargement in the 'Doctrine.' He does not say: Here is the original of Stirling's Theorem, here is the first appearance of the normal curve, here De Moivre anticipated Laplace as the latter anticipated Gauss. He does not even refer to the manner in which De Moivre expanded the Newtonian theology and directed statistics into the channel down which it flowed for nearly a century. Almost everywhere in his 'History' Todhunter seizes a small bit of algebra out of a really important memoir and often speaks of it as a school exercise, whereas the memoir may have exerted by the principles involved a really wide influence on the development of the mathematical theory of statistics, and ultimately on statistical practice also. Todhunter fails almost entirely to catch the drift of scientific evolution, or to treat that evolution in relation to the current thought of the day, which influences science as much as science influences general thought. The causes which led De Moivre to his 'Approximatio' or Bayes to his theorem were more theological and sociological than purely mathematical, and until one recognises that the post-Newtonian English mathematicians were more influenced by Newton's theology than by his mathematics, the history of science in the eighteenth century—in particular that of the scientists who were members of the Royal Society—must remain obscure."

Minor Contributions of Various Writers. There is no reason to believe that De Moivre's statement of the equation for the curve of facility of errors attracted any attention, and several other writers, evidently ignorant of his work, offered hypotheses as to the shape of this distribution. Simpson[41] suggested two hypotheses, that the distribution of errors might be a rectangle, each magnitude of error being equally probable, or that it might be an equilateral triangle, the probability of an error being inversely proportional to its magnitude.[42] Daniel Bernoulli thought that a circle would represent the distribution of chance errors, and called it the *circulus moder-*

[40] Communication in *Nature*, CXVII, 552 (April 17, 1926).

[41] Thomas Simpson (1710–1761) was a somewhat erratic and irascible genius who wrote on calculus, algebra, probability, and other mathematical subjects, and taught in the Woolwich Military Academy for some years. He was one of the most important of English mathematicians in the relatively sterile period following the death of Newton. For further details see Frances Marguerite Clarke, *Thomas Simpson and His Times*, (Baltimore, 1929).

[42] "A Letter to the Right Honorable George Earl of Macclesfield, President of the Royal Society, on the Advantage of Taking the Mean of a number of Observations, in practical Astronomy," *Philosophical Transactions*, XLIX (1755), Part I, 82–93. This was reprinted in 1757 with some omissions and some new matter, in *Miscellaneous Tracts.*

ator.[43] Laplace proposed for the law of facility of errors more than one formula which he later abandoned. In 1774 he suggested the equation $\phi(x) = \frac{1}{2}m\, e^{-mx}$, x being considered positive,[44] and in 1778 he said that the law of facility of error[45] should be $\phi(x) = \frac{1}{2a} \log_e \frac{x}{a}$.

3. THE WORK OF THE MATHEMATICAL ASTRONOMERS

New Interest in Probability. Many of the mathematicians in the 18th century wrote one or more memoirs dealing with probability, but it was not a period of great mathematical activity and no important advances in the theory were made until near the close of the century. The two decades just after 1800, however, saw the law of facility of error established on scientific principles, the foundation laid for a theory of errors of observation, and the new discoveries widely disseminated among European astronomers. All this was due to the work of the two greatest mathematicians then living, Laplace[46] and Gauss.[47] The discoveries of these two men and the work of Legendre on Least Squares created much excitement among the astronomers, and memoir followed memoir in rapid succession. Many

[43] "Velim ante omnia, ut quisque obseruator probe secum perpendat atque aestimat maximum errorem, quem nunquam se transgressurum, quotiscunque obseruuationem repetet, moraliter certus sit, si ne vel omnes Deos Deasque offendat iratas; sit ipsemet dexteritatis suae index nec seuerus nec blandus. Nec tamen admodum multum refert siue congruum siue quodammodo temerar ac de re tulerit indicium; tum radium *circuli moderatoris* faciat maximo errori praememorato aequalem." "Diiudicatio maxime probabilis plurium obseruationum discrepantium atque verisimillima inductio inde formanda," *Acta Academiae Scientiarum Imperialis Petropolitanae pro Anno MDCCLXXVII, Pars Prior*, I, 3–23 (St. Petersburg, 1778).

[44] "Determiner le milieu que l'on doit prendre entre trois observations données d'un même phénomène," *Mémoires de Mathématique et de Physique, Presentées à l'Académie Royale des Sciences, par divers Savans* (1774).

[45] "Mémoire sur les Probabilités," *Histoire de l'Académie Royale des Sciences, pour l'Année MDCCLXXVIII* (1781). See §XIII, 259.

[46] Pierre-Simon, Marquis de Laplace (1749–1827) wrote on probability, the calculus, differential equations, geodesy, and celestial mechanics.

[47] Carl Friedrich Gauss (1777–1855) was, like Laplace, born in very humble circumstances, but he became the greatest of German mathematicians. He wrote important works on the theory of numbers, celestial mechanics, geodesy, and least squares, and contributed to practically every known branch of mathematics. Probably the first measurement of a reaction time was one made at his suggestion in 1838. (Scripture, *Experimental Psychology, Instructor's Manual*, p. 443.) Smith (*History*, I, 504) says that with the physicist Wilhelm Weber he laid the foundations for telegraphy. This connection with Weber is worthy of note in view of the latter's connection with Fechner. (Weber's Law is named for Ernst Heinrich Weber, not Wilhelm Weber. The two were brothers.)

of these begin with a statement that the ideas contained in the paper are chiefly due to "that celebrated geometer, M. Laplace," or to "our Gauss," or to both. Among the men who wrote noteworthy articles on the subject of probability, the law of error, and the applications of this law to the variable results of observations, before 1850, were the following: in Germany—Gauss, Bessel, Encke, Littrow, and Hagen; in France—Laplace, Legendre, Poisson, Fourier, Cauchy, Puissant, Bienaymé, and Bravais; in England—Ivory, De Morgan, Galloway; in Belgium—Quetelet; in Italy —Plana; in America—Adrain; in Russia—Tchebycheff.

In 1778, in the memoir containing the second untenable hypothesis as to the shape of the curve of error, Laplace evaluated the integral of e^{-t^2} from zero to infinity (p. 293). This appears to argue a recognition of the existence of that equation which we now call the law of error, and should probably be considered the first statement of it after the time of De Moivre. In 1783 Laplace suggested that the integral of e^{-t^2} was encountered so frequently in calculus that a table of the values of the integral would be extremely useful, and he gave an approximation formula for making the computations.[48] In 1799 Kramp published the first tables of the probability integral in a book on refractions.[49]

Contribution by an American. The next proof of the law of errors was made by the American mathematician Adrain.[50] In 1808 he published two proofs of this law in a brief paper entitled "Research concerning the

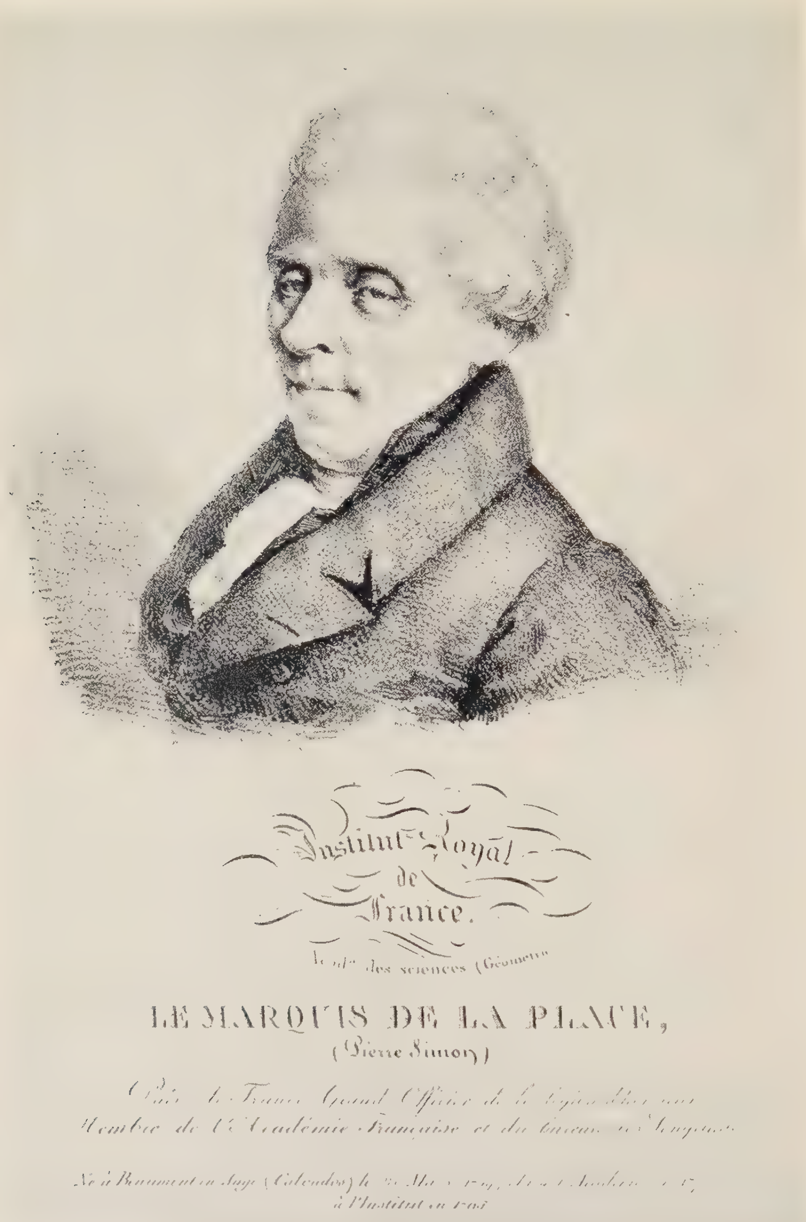
[48] "L'intégrale $\int dt \cdot e^{-t^2}$ se rencontre fréquement dans cette analyse, & par cette raison, il-seroit très-utile de former une table de ses valeurs depuis $t = \infty$, jusqu'à $t = 0$. Lorsque cette intégrale est prise depuis $t = T$, jusquà' $t = \infty$, T étant égal ou plus grand que 3 on pourra faire usage de la formule

$$\int dt \cdot e^{-t^2} = \frac{e^{-T^2}}{2T} \cdot \left(1 - \frac{1}{2T^2} + \frac{1.3}{4T^4} - \frac{1.3.5}{8T^6} + \&c.\right);$$

qui donnera une valeur alternativement plus grande & plus petite que la véritable." "Suite du Mémoire sur les approximations des Formules qui sont fonctions de très-grands Nombres," *Histoire de l'Acadêmie Royale des Sciences, Annêe MDCCLXXXIII* (1786), p. 434.

[49] See page 58.

[50] Robert Adrain (1775–1843) was a native of Carrickfergus, Ireland, and came to America after being shot in a battle with the forces of the crown. He taught mathematics in Rutgers—then called Queen's College, in Columbia College, and in the University of Pennsylvania. He is said to have resigned from the latter position after disciplinary troubles in his classes. For further biographical details, see Coolidge, "Robert Adrain, and the Beginnings of American Mathematics," *American Mathematical Monthly*, XXXIII (1926), 61–76.



PIERRE-SIMON, MARQUIS DE LAPLACE (1749–1827)
From the collection of Professor David Eugene Smith

probabilities of the errors which happen in making observations &c."[51] The first proof is based on the very arbitrary assumption that the errors in measured quantities are proportional to the quantities themselves. The second proof is noteworthy as being apparently the first study of the probability that two errors will occur together.[52]

Laplace. Most of the numerous memoirs on probability written by Laplace were finally brought together and reprinted, with some new material, in *Théorie Analytique des Probabilités*.[53] This is generally considered the greatest single work on the subject of probability. The extreme difficulty of Laplace's style of composition has been mentioned by almost every commentator. De Morgan[54] says it is "by very much the most difficult mathematical work we have met with,"[55] and Ellis,[56] "It must be admitted that there are few mathematical investigations less inviting than the fourth chapter of the Theorie des Probabilites, which is that in which the method of least squares is proved." In his *History of the Theory of Probability*, Todhunter devotes pages 464–613 to Laplace, reproducing a considerable part of his analysis. Speaking of his proof of the method of least squares, Merriman says: "The principal objection against the validity of the proof is that it requires an infinite or very large number of observations. With this requirement, however, Gauss's proof of 1809 becomes perfectly logical. . . . . Laplace's proof has been greatly improved by subsequent writers. Ellis in 1844 extended it to any number of unknown quantities, Todhunter

[51] *The Analyst or Mathematical Museum*, I (Philadelphia). This magazine was edited by Adrain. It had a brief existence, one volume only appearing. This single volume is now comparatively rare, but through the courtesy of the Yale library I have been able to consult it. The paper was reprinted in part in 1871 by Abbe in "A Historical Note on the Method of Least Squares," *American Journal of Science*, 3rd series, I, 411–415. The second proof is reproduced in Merriman's "List of Writings relating to the Method of Least Squares," and is also treated by Walker in "The Relation of Plana and Bravais to Correlation Theory," *Isis*, X (1928), 466–484. The assumptions underlying the first proof were criticised by Glaisher in his paper "On the Law of Facility of Errors of Observations and on the Method of Least Squares," *Memoirs of the Royal Astronomical Society*, XXXIX (1872), 75–124.

[52] See page 94 for the relation of this to correlation theory.

[53] 1st ed. 1812; 3rd ed. 1820; 4to, pp. cxliii + 506, with 3 supplements; 4th ed. 1847, 4to, pp. cxcv + 691.

[54] Article on probability in the *Encyclopaedia Metropolitana*.

[55] Although some portions of the *Théorie Analytique* have been read, most of the statements concerning Laplace are taken from secondary sources.

[56] "On the Method of Least Squares," *Cambridge Philosophical Transactions* VIII, Part I (1844), 204–240.

in his *History of Probability*, pp. 560–588, supplied a valuable commentary, and Glaisher in 1872 presented it in a clear and simple form. See also . . . . 1824 Poisson, 1847 De Morgan, 1852 Bienaymé, 1861 Airy, 1873 Laurent and 1875 Dienger."

Two years after the publication of *Théorie Analytique* Laplace brought out *Essai philosophique sur les probabilités* (1814). At least one English[57] and two German editions have appeared, and the work had a powerful effect on the thinking of Europe.[58] Both of these works of Laplace are quoted extensively by almost every writer on probability since that time. Soon after their publication there appeared, in various languages, a large number of books and papers whose purpose was either to make the works of Laplace available to persons who could not read French, or so to simplify and organize his theory that it could be read by a larger number of persons.

Gauss. Acknowledging his debt to Laplace,[59] Gauss made use of the law of probability in Section III of Book II of his great work on the theory of the motions on the heavenly bodies,[60] and also reached the principle of least squares, which had first been published by Legendre in 1805. This section of the book relates to the determination of the orbit of a planet from any number of observations. Gauss does not seem to have intended to establish a principle applicable to other types of investigation. The points most significant for modern statistics are these:

1. The definition of h as the measure of precision, h being equal to $\frac{1}{\sigma\sqrt{2}}$, in modern symbols.

2. The enunciation of the principle of least squares (p. 260 in the English version).

3. The general idea that among a system of conflicting values obtained

[57] *A Philosophic Essay on Probabilities*, translated from the 6th French edition by Truescott and Emory, New York, 1902.

[58] See the reference to Laplace in Keynes's *Treatise on Probability* (London, 1921, 466 pp.).

[59] " . . . . et quum per theorema elegans primo ab ill. Laplace inuentum, integrale $\int e^{-hh\Delta\Delta}d\Delta$, $a\ \Delta = -\infty$ usque ad $\Delta = +\infty$, fiat $\frac{\sqrt{\pi}}{h}$, (denotando per π semicircumferentiam circuli cuius radius 1) functio nota fiet $\phi\Delta = \frac{h}{\sqrt{\pi}}e^{-hh\Delta\Delta}$." *Theoria Motus Corporum Coelestium in Sectionibus Coniciis Solem Ambientium*, Book II, Section III, § 177 (Hamburg, 1809).

[60] *Theoria Motus* was translated in 1857 by Charles Henry Davis of the U. S. Navy.

Carl Friedrich Gauss (1777–1855)
From the collection of Professor David Eugene Smith

from measurements none of which have perfect accuracy, there may exist a most probable value which can be found by numerical methods.

4. The statement that the standard deviation of a mean is the standard deviation of the distribution divided by the square root of the number of cases.[61]

Gauss's approach to the normal curve is of enough interest to quote at some length. "The investigation of an orbit having, strictly speaking, the *maximum* probability, will depend upon a knowledge of the law according to which the probability of errors decreases as the errors increase in magnitude: but that depends upon so many vague and doubtful considerations—physiological included—which cannot be subjected to calculation, that it is scarcely, and indeed less than scarcely, possible to assign properly a law of this kind in any case of practical astronomy. Nevertheless, an investigation of the connection between this law and the most probable orbit, which we will undertake in its utmost generality, is not to be regarded as by any means a barren speculation." Let $\phi\Delta$ be the probability to be assigned to each error Δ. "Now although we cannot precisely assign the form of this function, we can at least affirm that it should be a maximum for $\Delta = 0$, equal, generally, for equal opposite values of Δ, and should vanish, if, for Δ is taken the greatest error, or a value greater than the greatest error." The probability that an error lies between D and D′ is given by the integral $\int \phi\Delta \cdot d\Delta$. Gauss then makes the further assumptions that an error is composed of a number of elementary errors, and that the arithmetic mean of the observed values is the most probable value of the measure. He sets up a differential equation and by integration finds

$$\log \phi\Delta = \tfrac{1}{2}k\Delta\Delta + \text{constant},$$

whence

$$\phi\Delta = Ke^{\frac{1}{2}k\Delta\Delta},$$

and K must be negative in order that the probability shall really have a maximum. Two years later Gauss published a memoir in which was used for the first time the notation

$$[ab] = a'\,b' + a''\,b'' + a'''\,b''' + \;.\;.\;.\;.\;.$$

so familiar in later works.[62]

[61] "Ceterum gradus praecisionis medio ita inuento attribuendus secundum principia mox explicanda erit $= \sqrt{(ee + e'e' + e''e'' + e'''e''' + \text{etc.})}$, ita ut quatuor vel nouem observationes aequa exactae requirantur, si medium praecisione dupla vel tripla guadere debet, et sic porro." Let $e = \dfrac{1}{\sigma\sqrt{2}}$. Then this radical is $\sqrt{\dfrac{N}{2\,\sigma^2}} = \dfrac{1}{\dfrac{\sigma\sqrt{2}}{\sqrt{N}}} = \dfrac{1}{e'}$, where e' is the degree of precision of the mean.

[62] Merriman, p. 166.

The work of Gauss on probability was scattered through numerous Latin and German memoirs, some of which dealt primarily with astronomy or other sciences, and was not brought together in a single volume such as Laplace's *Théorie Analytique.* It did not rest on such broad principles or provide so complete a theoretic basis for the study of mathematical probability as did Laplace's, but it established the theory of errors of observation, of which modern statistical theory is the direct descendant. Gauss was much concerned with the practical as well as the theoretical problems of astronomy, and these included the problem of dealing with the variable results of observations. Indeed it was to his interest in the individual irregularites of observers that Scripture credits the first psychophysical measurement of a reaction time. "About 1838 the personal equation began to receive regular notice in astronomical observations, as appears from the publications of Airy and Gerling of that year. It was natural to wish for a comparison of the astronomer's record with the real time of transit. At the suggestion of Gauss an artificial transit was arranged by Gerling, the object observed being a slow pendulum. This is probably the first measurement of a reaction time."[63] Because of this practical concern for measuring variability, Gauss developed a very large body of material related to probable errors, and a great many of the formulas in common use today are to be found in his writings or in those of the contemporary German astronomers Encke and Bessel who followed him rather closely.

First Use of the Term Probable Error. In 1815 Bessel[64] used the term *probable error* (*der wahrscheinliche Fehler*) for the first time.[65] The next year Gauss published a memoir on the determination of the accuracy of observations[66] in which he says that by the method of least squares the probability of errors of observation is given by the formula $\frac{h}{\sqrt{\pi}} e^{-hh}$ where $h = \frac{1}{\sigma\sqrt{2}}$, and that $\frac{\rho}{h}$ is the *wahrscheinlicher Fehler*, which he designates as r.

[63] E. W. Scripture, *The New Psychology*, p. 443 (1897).

[64] Friedrich Wilhelm Bessel (1784–1846) was one of the greatest astronomers of his time, and also a physicist of note.

[65] "Ueber den Ort des Polarsterns," *Berliner Astronomisches Jahrbuch für 1818*, pp. 233–240. The statement that this is the first use of the term is due to Merriman. I have seen the term in papers written by Gauss and Bessel in 1816, and have not found it before 1815. For the exact words used by Bessel in introducing this term, see page 51.

[66] "Bestimmung der Genauigkeit der Beobachtungen," *Zeitschrift für Astronomie und verwandte Wissenschaften*, I, 187–197.

He gives a small table of nine rows, adopted from Kramp's tables, showing the probability that an error will not deviate from the mean by more than a certain amount. The table is arranged to express deviations either in terms of $\sigma\sqrt{2}$ or in terms of the probable error. In this paper he also has a treatment of a topic closely related to the Pearson treatment of moments (See "Moments") and suggests the computation of the probable error by a method which gives what we would now call the median deviation. He says that the probable error of this "median deviation" is $\frac{0.7520974}{\sqrt{N}}$. (See "Median.")

Practical Rules for Computation. The theory of probability as applied to errors of observations and practical directions for computation were set forth in a work by Encke[67] on the method of least squares, in 1832.[68] Except for the absence of the word "statistics" this might almost be a text in the mathematical theory of statistics, and reads much as though it had come from the Pearson laboratory in an early day. The style is simple, clear, readable. Encke says that there is nothing of consequence in the book except what he has taken from Gauss, but that, as he has occasionally changed the form, he does not give any specific references to Gauss's works. All of the following formulas are found in this work:

1. The formula for the standard error of a mean.
2. The formula for the standard error of a standard deviation.
3. The "gross-score formula" for computing a standard deviation.[69]
4. The formula for deriving the probable error from the standard deviation. He wrote $r = 0.674489\ \varepsilon_2$.
5. The formula for computing a standard deviation from any arbitrary origin.

[67] Johann Franz Encke (1791–1865) became a student of Gauss at the age of twenty. The following year he made astronomical computations which Gauss thought worthy of publication. Encke was for many years the editor of the *Berliner Astronomisches Jahrbuch*, and an astronomer of distinction. For a full biography, see Bruhns, *Johann Franz Encke, sein Leben und Wirken*, Leipzig, 1869.

[68] "Über die Methode der kleinsten Quadrate," *Berliner Astronomisches Jahrbuch für 1834*, pp. 249–312.

[69] Encke wrote
$$\varepsilon_2 = \sqrt{\frac{[nn] - \frac{[n]^2}{m}}{m}}.$$

Compare the formula
$$\sigma = \sqrt{\frac{\Sigma X^2 - \frac{(\Sigma X)^2}{N}}{N}}.$$

6. Recursion formulas for the moments of the curve of errors, and the numerical values of the first nine moments.[70]

7. The probable error for the kth root of the kth moment up to $k = 6$, and the number of observations needed to determine each with a degree of accuracy equal to the accuracy of the middle error (standard deviation).

8. The probable error of a sum (of independent variables).

9. The probable error of a weighted sum (of independent variables).

10. A method for computing the probable error similar to the method of finding a quartile.

The book served as the basis for numerous textbooks. Much of this material is embodied in Airy's *Combination of Observations* (1861). At the time of its publication, this and the two other memoirs in the series, must have been invaluable, and even now are well worth reading.

Simplification of Theory. One of the first attempts to reduce the theory of probability to rules of thumb which might be followed by persons having no command of mathematics beyond simple arithmetic was made by De Morgan[71] in *An Essay on Probabilities and on Their Application to Life Contingencies and Insurance Offices.*[72] The author says that heretofore the work in probabilities has been possible only to the expert mathematician, but that "those who already admit that the theory of probabilities is a desirable study, must of course allow that persons who cannot pay much attention to mathematics, are benefited by the possession of rules which will enable them to obtain at least the results of complicated problems, and which will therefore, permit them to extend their inquiries further than a few simple cases connected with gambling." (p. viii, preface.) In spirit the book is closely akin to Thorndike's *Mental and Social Measurements*, and in its day must have been extremely useful. It is really a handbook for "an experimental philosopher requiring only a sufficient practical rule for the treatment of a set of observations." It is full of practical advice to persons tempted to draw unwarranted conclusions from a situation, the applications being chiefly to life-insurance, games of chance, and lotteries.

[70] This statement requires some qualifications. See page 72 for further treatment.

[71] Augustus de Morgan (1806–1871), brilliant and somewhat erratic, wrote on a wide range of mathematical topics. His *Budget of Paradoxes* (1st edition brought out by Mrs. de Morgan in 1872, 2nd edition by D. E. Smith in 1915) is one of his best known works. He was the greatest English authority of his day on the theory of probability and his scholarly paper on that subject in the *Encyclopaedia Metropolitana* is still a classic. For further biographical information, see *Memoir of Augustus De Morgan* by Sophie Elizabeth De Morgan, London, 1872. Todhunter was one of De Morgan's students.

[72] Published in Lardner's *Cabinet Cyclopaedia*, September 1838.

His minute directions for the steps to be taken in constructing the normal curve remind one of the "job analysis" blanks for statistical computations now on the market. Of this curve he says, "To this table (of ordinates of the curve) I shall have continual occasion to refer: into it, in fact, is condensed almost the whole use I shall have to make of the higher mathematics." This sounds remarkably like some of our texts in educational statistics.

About the same time, De Morgan wrote another article on the "Theory of Probabilities"[73] in which he presented a summary of the mathematical theory of probability and a survey of its development. Here he discusses the method of least squares, the application of the calculus of definite integrals, and some complicated problems in probability. One portion of the essay which is of paramount interest because it indicates the direction in which the minds of scientific men were then turning,[74] is the section dealing with the probabilities of testimony, of miracles, of moral questions, of decisions of a jury. He even suggests that an investigation into the validity of the English practice of requiring a unanimous decision from a jury might be made by studying five hundred trials in which the jury have delivered their verdict at once to see if they show a smaller percentage of error than five hundred trials in which the jury deliberated for two hours or more. Truly this has a modern flavor.

In this connection it must be remembered that in 1833 Quetelet had been in Cambridge as official Belgian delegate to the British Association, and had been influential in having a statistical section formed, and that when the Royal Statistical Society of London was formed in 1834, Quetelet was elected a corresponding member. Whether De Morgan and Quetelet had met in person or not, each knew the work of the other. In Mrs. DeMorgan's *Memoir of Augustus De Morgan* is a letter from Quetelet to De Morgan thanking him for sending a table[75] and a copy of his work which

[73] *Encyclopaedia Metropolitana.* No data is given, but Sotheran ascribes it to 1836. Mrs. De Morgan speaks of the paper as originally published in January, 1838. The article was republished in *Mathematical Papers of De Morgan*, pp. 393–490.

[74] Compare a paper by John Tozer, Barrister-at-law, "On the Measure of the Force of Testimony in Cases of Legal Evidence," *Cambridge Philosophical Transactions*, VIII, Part II (1844), 143–158. The writer assigns numerical values to the probabilities of the accuracy and sincerity of witnesses, and by elaborate mathematical processes attempts to reach a numerical measure of the probability that a given defendent is guilty or innocent. Laplace had discussed the probability of testimony in his *Théorie Analytique*, pages 446–461 in the edition of 1820, and had assumed that *a priori* probabilities could be assigned in this field to justify theoretical problems like those of drawing balls from an urn.

[75] Could this be a table of the probability integral?

Quetelet says had given him great pleasure. Apparently the two men had been introduced by Babbage, but had not met at this time. Mrs. De Morgan thinks that her husband had met Quetelet in Paris in 1830, and that the latter had not remembered this when he wrote.[76]

It was also about this time that Poisson was writing on the application of the theory of probability to testimony and the decisions of law courts.[77]

4. APPLICATION OF THE LAW OF ERROR TO SOCIAL PHENOMENA

Early Attempts. We have seen that by 1812 the mathematical theory of probabilities, including that particular law of facility of errors which we now term the *normal law,* was well established on broad scientific principles, and that in one or two decades more the law of errors had come into very general use among astronomers and physicists, and was occasionally employed in mathematical problems relating to artillery fire. As yet, however, there had been no successful attempt to connect the theory of probability with social phenomena. There had been, it is true, several incipient ventures in that direction. Several writers had discussed the constancy of the ratio of male to female births, usually drawing from their investigations more of theological speculation than of sociological fact.[78] The

[76] "Mon cher Monsieur,—Je vous remercie beaucoup pour l'obligeance que vous avez eue de m'addresser la table que vous m'avez promise, et vos ouvrages que j'ai parcourus déjà avec le plus grand plaisir. La méthode que j'ai trouvée dans vos livres élémentaires augmente encore le prix que j'attache aux suffrages honorables que vous avez bien vouler exprimer pour les miens.

"Je suis très charmé que notre ami commun, M. Babbage, m'ait procuré le plaisir de faire votre connaissance: je désire beaucoup de cultiver. Je regrette de ne pouvoir aller moi-même vous exprimer mes remercîments, mais, comme je vais aujourd'hui, j'ai dû me borner à vous écrire, comptant bien sur votre indulgence.

"Recevez, je vous prie, mon cher monsieur, l'expression de mes sentimens distingúes.

Tout à vous,

Quêtelet."

[77] Poisson, *Recherches sur la probabilité des jugements en matière criminelle et en matière civile, précédées des règles générales du calcul des probabilités* (Paris, 1837).

[78] John Arbuthnot, "An Argument for Divine Providence, taken from the constant Regularity observ'd in the Births of both Sexes," *Philosophical Transactions*, XXVII, 186–190.

S'Gravesande, *Œuvres Philosophiques et Mathématiques* (Amsterdam, 1774).

Nicolas Bernoulli, several letters. See comment by De Moivre in preface to 3rd edition of *The Doctrine of Chances*, and Todhunter, *History*, pp. 197–198.

Daniel Bernoulli, "Mensura Sortis ad fortuitam successionem rerum naturaliter contingentium applicata," *Novi Commentarii* of the Petersburg Academy, XIV. (Reference due to Todhunter.)

De Moivre, *Doctrine of Chances*, 3rd ed.

advantages of inoculation as a preventive of smallpox[79] had been discussed by Daniel Bernoulli[80] who attempted to solve the problem by calculations based on *a priori* assumptions regarding the prevalence and fatality of the disease. The same problem was considered by Trembley[81] in 1796. Bernoulli's methods and conclusions were attacked by d'Alembert[82] in several memoirs[83] but to either writer a statistical investigation was impossible because no long series of observations had been made. As has been already mentioned, Jacques Bernoulli had been convinced that the mathematical theory of probability contained far-reaching implications for the study of a wide variety of social matters, but had failed to present a cogent argument for his belief. While recognizing that his death occurred before this portion of the *Ars Conjectandi* was completed, and also recognizing that one of his biographers has said that the latter part of his life was handicapped by illness, we are still of the opinion that a man who had spent twenty years of study on probabilities and had produced no social applications more useful than these vague suggestions would have been most unlikely to succeed later in establishing a union of mathematical theory and social or economic investigation. Bernoulli does not appear to have been possessed by the passion for collecting data, for weighing and measuring and counting, which marked Quetelet and Galton.

Laplace, various memoirs, and passages in *Théorie Analytique des Probabilités.*

Trembley. (See Todhunter, *History*, p. 415.)

Derham, *Physico-Theology* (1713).

Süssmilch, *Die Göttliche Ordnung in den Veränderungen des menschlichen Geschlechts, aus der Geburt, dem Tode, und der Fortpflanzen desselben* (1761), I, 576 p. + 114 p. tables; II 625 p. + 78 p. tables.

Graunt, *Observations on the London Bills of Mortality* (1662).

[79] "Essai d'une nouvelle analyse de la mortalité causée par la petite Vérole, et des avantages de l'Inoculation pour la prévenir."

[80] Daniel (I) Bernoulli, nephew of Jacques and son of Jean (I) Bernoulli, was born at Groningen in 1700 and died at Basel in 1782. He taught mathematics in the Academy of St. Petersburg and in the University at Basel, and published some books and many memoirs. Most of his work on probability appeared in the memoirs of the Royal Academy of Sciences at St. Petersburg.

[81] Jean Trembley (1749–1811), born in Geneva, wrote chiefly on the calculus and on probabilities.

[82] Jean Baptiste le Rond d'Alembert (1717–1783). For interesting personal details, see Smith, *History*, I, 479–480.

[83] "Sur l'application du Calcul des Probabilités à l'inoculation de la petite Vérole," *Opuscules Mathématiques*, II, (1761), 26–95.

"Sur un Mémoire de M. Bernoulli concernant l'Inoculation," *Opuscules*, IV, 98–105.

"Sur les calculs relatifs à l'Inoculation," *Opuscules*, IV, 310–341, and probably other memoirs.

As early as 1699, an anonymous essay[84] had attempted to apply the theory of probabilities to the credibility of testimony, and similar speculations were later put forward by Nicolas Bernoulli,[85] Condorcet,[86] and others. Voltaire[87] also wrote an essay on probabilities as applied to courts of law in which he said that almost everything in human life could be resolved into a problem in probability.[88] Condorcet had also expressed a belief that the truths of moral and political science could be determined with the same certitude as those of physical science, of which some branches such as astronomy appeared to approach mathematical exactness.[89] The applications of probability to mortality tables and the computation of annuities had received more attention than any other economic aspect of the theory, and even as early as 1776 there was published a list of sixty-two titles dealing with these topics.[90] Here we find, among others, the well known names of Graunt, Petty, Halley, Simpson, Hodgson, De Moivre, Hume, Davenant, and Price from England, of d'Alembert, Deparcieux, and Buffon from France, of Keerseboom, Struyk, and Schwenke from Holland, of Euler, Daniel Bernoulli,[91] and Süssmilch from Germany, of Wargentin from Sweden, and other less familiar names from these countries and from Italy.

[84] "A Calculation of the Credibility of Human Testimony," *Philosophical Transactions*, XXI, 359–365. Lubbock and Bethune ascribe it to Craig, who published in that year *Theologiae Christianae Principia Mathematica* a book of somewhat similar nature.

[85] Nicolas (I) Bernoulli (1687–1759), nephew of Jacques Bernoulli and editor of his *Ars Conjectandi* (1713), was a professor of mathematics at Padua and later at Basel. He had had legal training, and his first mathematical writing related to the application of the theory of probability to the law, *De Usu Artis Conjectandi in Jure*, Basel, 1709.

[86] Marie-Jean-Antoine-Nicolas Caritat, Marquis de Condorcet (1743–1794), a scholarly writer on calculus, differential equations and the theory of probability, ended his life by taking poison during the Reign of Terror.

[87] François Marie Arouet (1694–1778).

[88] "Presque toute la vie humaine roule sur des probabilités. Toute ce qui n'est pas démontré aux yeux, ou reconnu pour vrai par les parties évidemment intéressées à le nier, n'est tout au plus que probable. . . . . Cependant il faut prendre un parti, & il ne faut pas le prendre au hazard. Il est donc nécessaire à notre nature faible, toujours sujette à l'erreur, d'étudier les probabilités avec autant de soin que nous aprenons l'arithmetique & la géométrie. Cette étude des probabilités est la science des Juges: science aussi respectable que leur autorité même, puisqu'elle est le fondament de leurs décisions. Un Juge passe sa vie à peser des probabilités les unes contre les autres, à les calculer, à evaleur leur force." *Essai sur les probabilités en fait de justice*, 1772, 34 pages.

[89] See Todhunter, *History*, p. 351.

[90] *La Dottrina degli Azzardi applicata ai Problemi Della Probabilità della Vita, Della Pensioni Vitalizie, Reversioni, Tontine, ec. Di Abramo Moivre* (Milan, 1776).

[91] *La Dottrina* places Bernoulli among the German writers, though at this time he was probably in Basel.

The Beginnings of Descriptive Statistics. Contemporaneous with the development of the theory of probabilities and yet almost entirely unrelated to it, there had been growing two other movements most important in the historical development of statistics. One of these was begun by John Graunt in 1662 and has culminated in the establishment of statistical bureaus with their comprehensive publications. The other movement, the growing importance of official statistics, had its beginnings as far back in antiquity as recorded history goes. David numbered his people,[92] the Egyptians, the Babylonians, and the Romans preserved records of the resources of the state.[93] All three of these tendencies—the mathematical theory of probabilities; the growth of agencies for enumeration, calculation, and measurement; and the rise of the modern state with its increasing needs for information concerning its own resources and those of its neighboring rivals—all of these must be acknowledged as belonging to the ancestry of modern statistical theory and practice.

About one hundred years ago, these three movements came to a focus in the person of a great Belgian who was at once a mathematician, astronomer, anthropometrist, supervisor of official statistics for his country, prime mover in the organization of a central commission on statistics, instigator of the first nation-wide census, ardent collector of statistics, university teacher, author of many books and papers, carrying on an extensive correspondence with most of the scholars of his day, and in his leisure hours a poet and a writer of operas. Before speaking further of Quetelet,[94] it seems necessary to trace briefly the course of development of official and descriptive statistics.

First Scientific Study of National Resources. The beginnings of the scientific statistics of national resources are placed by Meitzen about the middle of the 16th century. The first comprehensive work of the type was Münster's[95] *Cosmographia* (1536 and 1544). This attempted to treat systematically the geography, history, political organization, social institu-

[92] II Samuel, 24.

[93] Fuller accounts may be found in the following:

John, *Geschichte der Statistik* (Stuttgart, 1884), xv + 376 p.

Meitzen, *Geschichte, Theorie und Technik der Statistik* (Berlin, 1886), ix + 214 p.

Meitzen, *History, Theory, and Technique of Statistics*, trans. by Falkner (Philadelphia, 1891).

[94] See page 39.

[95] Sebastian Münster (1489–1552), though a Franciscan, became after 1529, Protestant professor at Heidelberg and Basel.

tions, commerce, and military strength of all the principal states then existing. Several other studies of similar nature are mentioned by John and Meitzen, but it was not until 1660 that statistical studies were first made a part of a university curriculum. Conring,[96] who began in that year to deliver a course of lectures on "Staatskunde" at the University of Helmstedt, has been called by John the father of statistics, a term more commonly reserved for Achenwall. Lectures similar to Conring's were delivered in the 17th century at the Universities of Geneva, Giessen, Jena, Frankfurt-on-Oder, and Halle.

The Word Statistics. The term *Statistik* occurs for what is probably the first time in the writings of Achenwall[97] who is usually hailed as the father of statistical science. Achenwall and those who followed his tradition—the men who would have been termed statisticians in the eighteenth century—did not devote themselves so much to enumeration and computation as to verbal descriptions of the political situation and of all facts of interest in their countries, and they looked with considerable unfriendliness and displeasure upon the "Table-statisticians."[98]

Early Statistical Compilations. The history of descriptive statistics, "Die Tabellen Statistik," had vague beginnings in the Middle Ages in certain rudimentary registrations of which the object was at first land, and later, the man power available for military purposes.[99] Pepin the Short

[96] Hermann Conring (1606–1681) was professor of medicine and the laws of nature, court physician to several princes, and author of scientific works on various subjects.

[97] Gottfried Achenwall (1719–1772) lectured at Marlborough and at Göttingen. His first statistical work was *Vorbereitung zur Staatswissenschaft der Europäischen Reiche* (1748). This was used later as the introduction to his *Abriss der Statswissenschaft der heutigen vornehmsten europäischen Reiche und Republiken* (1749). The term *Statistik* occurs in the preface.

[98] For the earlier use of *statist*, *statista*, *statisticum*, and similar terms, see the paper "The term 'Statistics.' Translated from a Work by Dr. V. John, Professor of the University of Berne, entitled 'Der Name Statistik—Eine Entomologisch-historische Skizze.' Berne, 1883." *Journal of the Royal Statistical Society* (1883), pp. 656–679. For an account of the early use of the term in English, see Yule, "Introduction of the words 'Statistics,' 'Statistical' into the English language," *Journal of the Royal Statistical Society*, LXVIII (1905), 391–396. Yule says that Zimmerman's *Political Survey of the Present State of Europe* (London, 1787) contained the sentence " . . . . this science, distinguished by the new found name of *statistics*, is become a favorite study in Germany."

[99] See an article by Fernand Faure, "The Development and Progress of Statistics in France," in *The History of Statistics, Their Development and Progress in Many Countries* (Compiled and edited by John Koren, New York, 1918).

(758) and Charlemagne (762) demanded detailed descriptions of church lands, while several works written in France during the first half of the ninth century gave a partial enumeration of the serfs attached to the land. The lists of taxes levied in Paris in 1292 and in 1300 furnished also a registration, street by street, of the artisans who were subject to the tax. A French document dating from about 1200 gives the figures to show how many soldiers Philip Augustus could obtain from the commoners and the communes. In England we have the Domesday Book[100] of 1086 and other similar surveys.[101] Industrial statistics had a beginning in the work of the medieval guilds. Faure says that to the corporation of gaugers " . . . . belongs the honor of having been for the first time especially called upon to perform a statistical function," when they were authorized to measure all corn sold in the Paris markets whenever the quantity exceeded a prescribed amount, and that they were customarily asked to make out memoranda of these transactions. These statistical functions were confirmed and amplified by various ordinances passed during the fourteenth and fifteenth centuries, and other corporations performed similar services in other fields. There is also indication[102] that customs duties were levied in France as early as 1324 and that in 1393 the number of butchers, butcher-shops, and the number of cattle delivered weekly were known for Paris. All this presupposes some sort of enumerations, although these were not methodically made and preserved.

The custom of registering deaths and marriages is found in certain parts of Burgundy early in the fourteenth century, and in 1406 the registration of baptisms is mentioned, very probably for the first time, in the records of the Bishop of Nantes. During the sixteenth century the registration of deaths, baptisms, and marriages was made obligatory for the French curates[103] and in England the Bills of Mortality were instituted by a terror-stricken sovereign during the period of the plague.[104] Toward the end of

[100] Domesdaeg = judgment day. Record was made by a local jury of the name of a place, the name of the holder, number of hides in the manor, number of villeins, cotters, serfs, and freemen, and the extent of wood, meadow, pasture, mills and fishponds.

[101] As, for example, the Exeter Domesday, Winston Domesday, and the Bolden Book.

[102] Faure, *op. cit.*, p. 231 *et seq.*

[103] The Ordinance of Villers-Cotterets (1539) required that curates keep registers of baptisms. The Council of Trent (1563) sanctioned this measure and extended it to include marriages. The Ordinance of Blois (1579) formulated rules for the keeping of registers of baptisms, marriages, and burials.

[104] The registration of deaths was begun by Henry VII in 1532, and soon the registration of baptisms by the parish clergy was added. During another outbreak of the plague, weekly bills of mortality were issued, and in 1629 a record of the sex of the deceased was added.

this century there appeared in France a curious document on the Secret of French Finances,[105] which suggests at once the beginning of a movement to utilise numerical data in determining state policies, the extreme paucity and unreliability of such data as were then available, and the secrecy with which they were commonly shrouded. The author divulges neither his real name nor the source of his data. This work has been called "one of the first methodical essays in statistics,"[106] and "the real point of departure of the history of French statistics,"[107] and the author has been spoken of as "the first who knew how to handle the instrument *par excellance* of political economy, statistics."[108] Faure, however, is of a different opinion, and calls this the work of a politician rather than of a statistician.

The Census. The first registrations of modern time sufficiently systematic to be considered as immediate precursors of the decennial census, took place in the early part of the seventeenth century. Some of the more important events relating to the enumerations of this period are tabulated below:

Canada:

1605—The first record of population in Canada relating to the founding of Port Royal (now Annapolis Royal).

1608—A record relating to the founding of Quebec.

1663—The population of New France recorded as 2500.

1666—The first census of modern times to be taken *by name*.[109]

Sweden:

1608—Beginning of the register of Trinity Church in Uppsala.

1686—Passage of an ecclesiastical law making it obligatory to keep parish registers with the records of marriages, births, including illegitimate births, baptisms, names of all persons buried in the parish cemetery, and the names of persons moving into or out of the parish.

Norway:

1662—Census of all adult males for military purposes.

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[105] *Le Secret des Finances de France découvert et départi en trois livres* (1581), by Nicolas Froumenteau.

[106] Maurice Block, *Traité Théorique et Pratique de Statistique* 2nd ed., p. 34 (Reference taken from Faure.)

[107] Baudrillart, *Des Théories Politiques et des Idées Économiques au XVIeme siècle*, p. 87, and Levasseur, *La Population Français*, Vol. I, p. 55. (Reference taken from Faure.)

[108] Espinas, *Histoire des Doctrines Economiques*, p. 166. (Reference taken from Faure.)

[109] E. H. Godfrey says this is "a date prior to any modern census, whether European or American" (*History of Statistics*, p. 179). The returns were fairly complete, giving data on population, sexes, families, conjugal condition, age, profession and trades, and they filled 154 pages. The original copy is now in the Archives of Paris, and a transcript in the Archives of Ottawa.

England:

1629—Distinction of sex made in the weekly Bills of Mortality for London.

1661–62—Graunt's *Observations on London Bills of Mortality.*

1693—Mortality tables published by Halley.

1696—The Inspector General of Imports and Exports authorized to keep records of trade.

1699—Sir William Petty's[110] *Essay on Political Arithmetick.*

France:

1577—Publication of Jean Bodin's *Les Six Livres de la Republique*, which appears to have had a profound influence on the statistical works of the seventeenth century.

1599—Sully made Superintendent of Finance; beginning of a period of reform in the system of accounts and of new and more careful investigations in the statistics of the realm.

1615—*Traité d'Économic Polytique*, by Anthoyne de Montchrétien.[111]

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[110] Sir William Petty (1623–1687) was a political economist, the physician-general to the army in Ireland, a French linguist, a member of the coterie of mathematicians who gathered at the house of Mersenne, one of the original members of the Royal Society, head of the commission which surveyed Ireland, maker of a complete map of Ireland, inventor of a manifold letter-writer, of a new kind of land carriage, of 'a wheel to ride upon,' double-keeled vessel, and other devices. An account of his somewhat turbulent life is found in the *Dictionary of National Biography*. He repeatedly urged the government to provide for the systematic collection of statistics. Among his publications are:

A Treatise of Taxes and Collections, 1662.

Another Essay in Political Arithmetick concerning the growth of the City of London: with the Periods, Causes, and Consequences thereof, 1683.

Two Essays in Political Arithmetick, concerning the People, Housing, Hospitals, &c. of London and Paris . . . . tending to prove that London hath more people than Paris and Rouen put together, 1686.

Five Essays in Political Arithmetick, 1687, (printed in French and English on opposite pages.)

Observations upon the Cities of London and Rome, 1687.

Political Arithmetick, or a Discourse concerning the extent and value of Lands, People, Buildings; Husbandry, Manufacture, Commerce, Fishery, Artizans, Seamen, Soldiers; Public Revenues, Interest, Taxes, London 1690.

[111] Faure sums up the new political philosophy in this book as follows: " . . . . The uses of the enumeration of subjects and of their revenues are infinite; they afford the means of insuring the defense of the country and the peopling of the colonies, of rendering more clear the juridic condition of individuals, of knowing . . . . (to what social rank each one belongs and what is his occupation), of driving out vagabonds, loafers, ruffians who live in the midst of respectable people, of providing for the just grievances of the poor against the rich, of laying and collecting equitably the 'thousand kinds of imposts' which existed then and which 'the ancients never knew,' of abolishing the extortions of the officials 'who distribute and equalize taxes, subsidies, and imposts,' and finally of 'putting an end to all rumors, appeasing all complaints, quieting all movements, suppressing all occasions for riot.' "

1663—Plans drawn up by Colbert for enumerations which would include all parts of the administration, commerce and manufactures.[112]

1670—First publication for Paris of the number of baptisms, births, deaths, and burials registered according to the Ordinances of 1539 and 1579, these facts having been previously regarded as state secrets.

1672—Inquiries concerning the causes of increase and decrease in population.

1698, 1699, 1700—Formulation of *Les Mémoires des Intendants* from a questionnaire, a work which became the basis of almost all later French works on population before the Revolution.

The first work to throw any real light on the regularity of social phenomena was Captain John Graunt's[113] *Observations on the London Bills of Mortality*, 1662. In 1671 Jan de Witt tried to determine scientifically what should be the purchase price of annuities, and in 1693 the astronomer Halley[114] published his mortality tables.[115] This was the real beginning of a theory of annuities, for the tables which had been previously published, had not been scientifically constructed.[116]

Sir William Petty's *Essays on Political Arithmetick* (1699) illustrate the great dearth of collected statistical data in his day. Desiring to reach truth by means of calculation, and lacking empirical facts upon which to base his calculations, he made use of *a priori* assumptions that lead to results highly entertaining to the modern reader. Not his conclusions, but his methods and his belief in the use of number, weight, and measure

[112] Faure says: "We may say that in the hands of Colbert, as in the hands of Sully, statistics were the essential instrument with which these two great ministers, sixty years apart, and in strikingly analogous circumstances, succeeded in reestablishing order in the public finances and prosperity in the national economy of France." *Op. cit.*, p. 249.

[113] Captain John Graunt (1620–74), friend of Sir William Petty, possessed a "most dextrous and incomparable facility in short-writing." The full title of his *Observations* was *Natural and Political Observations mentioned in a following Index, and made upon the Bills of Mortality, by John Graunt, citizen of London. With reference to the Government, Religion, Trade, Growth, Ayre, Diseases, and the several Changes of the said City.* Though a shop-keeper, Graunt was admitted to the Royal Society immediately after the appearance of this work. Six editions appeared: (1) London, 1661, (2) London, 1662, (3) 1665, published by the Royal Society, (4) Oxford, 1665, (5) 1676, enlarged edition edited by Sir William Petty, (6) 1749, reprinted by Dr. Thomas Birch.

[114] Edmund Halley (1656–1742), friend of Newton, became astronomer royal of England in 1721. He wrote on astronomy, geometry, algebra, and conics, and computed logarithms and mortality tables. He was the first to compute the period of a comet and to predict the date of its return.

[115] "An Estimate of the Degrees of the Mortality of Mankind, drawn from Curious Tables of the Births and Funerals at the city of Breslaw; with an attempt to ascertain the Price of Annuities upon Lives," *Philosophical Transactions*, XVII (1693), 596–610.

[116] See Lubbock and Bethune, p. 45.

to disclose knowledge, are significant. The "art of reasoning by figures upon things relating to government"[117] was pursued by Davenant, Derham,[118] Süssmilch,[119] and many others. From these beginnings there developed insurance societies and all those various agencies, public and private, which devote themselves to the collection, tabulation, and publication of numerical data.[120]

The social upheavals which marked the close of the eighteenth century and the early years of the nineteenth, with their new emphasis upon the importance of the masses, doubtless had much to do with the great advances which were made about that time in the collection of population statistics. The Articles of Confederation provided for the taking of a triennial census, the Constitution of the United States changing this to a decennial census which should serve as the basis for apportioning representation in Congress. The first such census was taken in 1790. In France, the Revolution had the effect of placing the public statistics in the hands of men like Lavoisier and Neufchâteau, who were conscientiously searching for facts and who would place these at the service of the public; of giving publicity to the results of statistical investigations formerly kept secret; and of producing a very large number of works relating to such investigations, particularly in the field of public finance. The Constituent Assembly passed a law calling for a general enumeration in 1791, but the first complete census was not successfully carried out until some decades later. In Holland a government census was conducted in 1795, and the first decennial census in 1828. Norway and Denmark had held a general enumeration in 1769, and the former began decennial registration in 1815. A general enumeration had been proposed in Sweden as early as 1728, and in 1756 the committee in charge of the tabular records was made a permanent body. This is probably the first instance of the creation of a permanent, separate statisti-

[117] Davenant, *Discourses on the Public Revenues, and on the Trade of England*, London, 1698, pt. I, "Discourse I," p. 2. (Reference quoted by Hankins.)

[118] *Physico-Theology* (1713) by William Derham (1657–1735) is "the substance of XVI sermons preached in St. Mary le Bow-church, London, at the Hon^ble^ Mr. Boyle's lectures, in the year 1711 and 1712." The book ran through at least 13 English editions, as well as one each in French, German, and Swedish.

[119] Johann Peter Süssmilch (1707–1767), a military chaplain, wrote *Betrachtungen über die göttliche Ordnung in den Veränderungen des menschlichen Geschlechts aus der Geburt, dem Tode, und der Fortpflanzung desselben erwiesen*, during a military campaign. The book is based upon the works of Petty, Graunt, Arbuthnot, King, Derham, and Niuwentyt. Meitzen says his averages are "substantially correct to the present day."

[120] For the history of this development, see Meitzen, *Geschichte*, and John, *Geschichte*.

cal authority. Thus it is not only in theoretical statistical work that the Scandinavian countries have been leaders. England instituted the decennial census in 1801. Belgium had no sooner achieved its independence from Holland by the Revolution of 1830 than it organized a general statistical bureau, which soon became a model for other states to pattern after.

Educational Statistics. One of the first collections of what might be called educational statistics, made by an educational agency on a large scale, was made in Norway in 1840, when the Ecclesiastical and Educational Department published extensive statistical tables concerning the condition of education at the end of the year 1827. This was continued until 1853, when the statistics of schools were made a regular part of the official statistics of the country.

Political Economy, Mathematics, and Government Statistics. Thus the three powerful ways of thinking which have recently united to establish a science of statistics were in the second half of last century still far apart. While as a rule the economic theorist, the mathematician, and the collector of public statistics had too little in common, it must be noted that a striking number of mathematical astronomers made contributions to population statistics not only through their work on the theory of probability, but also by pratical assistance with the census or with the interpretation of the data derived from it. Experience in making and recording meteorological and magnetic observations, in formulating the laws of barometric pressure, intensity of atmospheric electricity and the like from a long series of records, of determining longitude and latitude, and of charting the courses of the heavenly bodies from a large number of observations which differed slightly among themselves, all this gave the astronomer an understanding of the significance of variable errors and a technique of dealing with mass phenomena which he often utilised for the public welfare in problems relating to population statistics. On the other hand "Political economy took a strong bias towards theory and abstract reasoning, and did not, to use a modern phrase, invite statistics to endorse the cheques drawn by speculation, an attitude which prevailed for nearly two generations, until exponents who were experts in mathematics as well as in economics entered the field.[121]

[121] Sir Athalstane Baines, formerly president of the Royal Statistical Society, in his article on the history of official statistics in Great Britain, in *The History of Statistics*, p. 368.

Quetelet. In order to understand how Quetelet[122] was able to take these three methods of research and to fuse them into one powerful tool for investigating social phenomena, we may profitably look briefly at his personal history.[123] It is Quetelet's use of the law of error which is the reason for introducing his name at this point, and we wish to see not only how he utilized that concept in a score of new connections, but also why his ideas were so widely disseminated. In spite of a natural preference for letters, Quetelet took up the study of mathematics and astronomy, and soon his interests extended to the whole field of science. He taught not only mathematics and astronomy, but also geodesy and physics and the history of the sciences. A statement which he made at the opening of this last-named course and often repeated afterward, reveals much of his manner of thinking:

"The more advanced the sciences have become, the more they have tended to enter the domain of mathematics, which is a sort of center towards which they converge. We can judge of the perfection to which a science has come by the facility more or less great, with which it may be approached by calculation."[124]

The building of the royal observatory at Brussels was almost entirely due to Quetelet's enthusiasm and persistence, and upon its completion he was made its director. In 1823 he spent three months in Paris studying astronomical instruments and methods, and while there formed friendships with Arago, Bouvard, Laplace, Poisson, Von Humboldt, Fresnel, and Fourier. He studied mathematics under Fourier and probabilities under Laplace, and from that time on his writings showed an increasing emphasis on the use of probabilities in scientific research. The plans for the census of 1829 were of his making, and the organization of the *Commission centrale de statistique* in 1841 was due to him. So efficient was the organization of this commission, of which he remained president until his death, that it served as pattern for other nations. It was at his suggestion that the London Statistical Society was formed in 1834, and at his instigation that

[122] Adolphe Quetelet (1796–1874) began to teach mathematics at the age of 17, in a private school. Later he taught in the College of Ghent, and in the Athenaeum at Brussels. At the age of 24 he was elected to membership in the Royal Academy at Brussels, of which he soon became the dominant personality.

[123] The chief source for this account is *Adolphe Quetelet as Statistician* (New York, 1908) by F. H. Hankins. He, in turn, has quoted largely from "Essai sur la vie et les ouvrages de Quételet," by Edward Mailly, one of Quetelet's students and his assistant for thirty-seven years. This is published in the *Annuaire de l'académie royale des sciences, des lettres et des beaux-arts de Belgique*, xli (Brussels, 1875), 109–297.

[124] Quoted by Mailly, p. 159, and by Hankins, p. 16.

the first International Statistical Congress met in Brussels in 1853. He was the first foreign member elected to the American Statistical Association after its organization in 1839. He was member of more than one hundred learned societies. He traveled in most of the European countries making personal friends of most of the great scientists then living and afterwards carrying on a voluminous correspondence with them.

For years he edited the *Correspondence mathématique et physique* to which leading scientists from all parts of Europe contributed. He labored untiringly for international cooperation in statistical matters, making the journey to St. Petersburg at the age of seventy-six to attend the international statistical congress, where he was the central figure. Moreover he wrote prodigiously, publishing his ideas on statistical methods again and again, and utilizing the same statistical technique no matter whether the investigation related to astronomy or to anthropometry, to meteorology or to morals, thus establishing the idea of a general method applicable to a wide range of subjects.

Quetelet made important contributions to the statistics of population, and effected a great improvement in the organization and administration of official statistics, and in the technique of statistical methods, but the two phases of his work which relate to the history of the normal curve are his application of the theory of errors to new fields and his concept of the "average man,"—"l'homme moyen."

Moral Statistics. "Quetelet's studies in moral statistics opened a new field to statistical research, the sphere of human actions, where all is apparently indeterminate and individual. His venture into this field created very wide discussion, especially in Germany . . . . and was significant in the development of the methods, concepts, and scope of statistics."[125] The most important of his memoirs on moral statistics, "Recherches sur la penchant au crime aux différens ages,"[126] makes a study of the influence of such factors as sex, age, education, climate, and seasons on criminal tendencies. Here, as always, he is impressed by the constancy of the number of crimes from year to year. "Thus we pass from one year to another with the sad perspective of seeing the same crimes reproduced in the same order and calling down the same punishments in the same proportions. Sad condition of humanity! . . . . We might enumerate in advance how many individuals will stain their hands in the blood of their fellows, how

[125] Hankins, *loc. cit.*, p. 83.

[126] *Nouveaux Mémoires de l'Académie des Sciences et Belles-Lettres de Bruxelles*, VII (1832), 87 p. Read July 9, 1831.

ADOLPHE QUETELET (1796–1874)
From the collection of Professor David Eugene Smith

many will be forgers, how many will be poisoners, almost we can enumerate in advance the births and deaths that should occur. There is a budget which we pay with a frightful regularity; it is that of prisons, chains and the scaffold."[127]

Educational Statistics Indebted to Quetelet. In a multitude of ways, current statistical investigations in the field of education are pursuing courses which Quetelet marked out nearly a century ago, or are completing tasks which he began. His contemporaries scoffed at him because he dared to suggest that a man's moral qualities might be measured by his actions and his intellectual vigor by his productions. They called him a materialist because he deduced from statistical documents facts which ran counter to the current theory of free-will. He suggested that instead of making numerous observations on an individual as he progressed through life, the changes from one age level to another might be studied by making observations on large numbers of people at different ages. He was convinced that the measurement of mental and moral traits waited only upon the collection of sufficient and trustworthy data, and that when such measurement was feasible, it would be found that the distributions of these traits conformed to the normal law.

Quetelet developed the conception of statistics as a general method of research applicable to any science of observation. He presented to the British Statistical Association at its meeting in Plymouth in 1841 a list of more than forty topics which he hoped might soon be investigated by statistical methods, the topics being grouped under the general headings of meteorology, physics, chemistry, botany, agriculture, zoology, and man.

Quetelet's influence upon anthropology was profound. From *Sur L'Homme* (1835) to *Anthropometrie* (1871) he worked incessantly on anthropometric measurements, and while his successors have sometimes disagreed with his conclusions or criticised his data, they have not been able to escape his general method. Here is one of the first points at which the work of Quetelet touches the American public school. The earliest statistical investigations of educational problems carried on in this country were largely the work of physicians and anthropologists, familiar with the labor of Quetelet.[128]

[127] Hankins, *loc. cit.*, pp. 55, 85.

[128] Note, for example, the following studies: H. P. Bowditch, "The Growth of Children," *Report of the Board of Health of Massachusetts*, Boston, 1877.

W. Townsend Porter, "On the Application to Individual School Children of the Mean Values Derived from Anthropological Measurements by the Generalizing Method," *Papers on Anthropometry*, Boston, 1894.

Some remarks made by Quetelet on the subject of the wide applicability of the curve of errors, inserted in the second volume of the *Bulletin of the Central Commission of Statistics of Belgium* lead to an exchange of letters[129] with Bravais[130] who questioned whether it ought to be assumed that the deviation of a physical quantity from its mean should necessarily be of this form. Although Quetelet discussed the possibility of asymmetrical distributions, the general effect of his writings was to stress the importance of the normal curve. Francis Galton, who was an admirer of the great Belgian and quoted him frequently, followed his lead in assigning to this law of distribution an important function in describing social phenomena. Thus he wrote in *Hereditary Genius*: "The Law is an exceedingly general one. M. Quetelet, the Astronomer Royal of Belgium, and the greatest authority on vital and social statistics, has largely used it in his inquiries. He has also constructed numerical tables, by which the necessary calculations can be easily made, whenever it is desired to have recourse to the law."

Florence Nightingale. The "Passionate Statistician"[131] was also an ardent admirer of Quetelet. In a letter to Galton[132] she says, "M. Quetelet gave me his *Physique Sociale* and his *Anthropometrie*. He said almost like Sir Isaac Newton: 'These are only a few pebbles picked up on the vast seashore of the ocean to be explored. Let the exploration be carried out.'

W. Townsend Porter, "The Growth of St. Louis School Children," *Transactions of the Academy of Science of St. Louis*, 1893.

W. Townsend Porter, "The Physical Basis of Precocity and Dullness," *Transactions of the Academy of Science of St. Louis*, 1893.

T. L. Bolton, "The Growth of Memory in School Children," *American Journal of Psychology*, IV (1891–92), 362–380. The study was made under the guidance of Boas, with data obtained from anthropometric measurements made on the children in the grammar schools of Worcester, Massachusetts, and measurements of hearing, eye-sight, and memory.

W. L. Bryan, "On the Development of Voluntary Motor Ability," *American Journal of Psychology*, V (1892), 123–204. This study was also largely due to Boas, and contained some original statistical theory.

W. L. Bryan, "School Experiments on Rate," *American Journal of Psychology*, 1892.

Measurement of 20,000 children in the public schools of Toronto in 1891 by A. F. Chamberlain, then Fellow in Anthropology at Clark University.

[129] *Lettres sur la Théorie des Probabilités Appliques aux Sciences Morales et Politiques*, Brussels, 1846; 412–424.

[130] See page 96.

[131] The phrase was used by Edward T. Cook in his *Life of Florence Nightingale*.

See E. W. Kopf, "Florence Nightingale as Statistician," *Publications of the American Statistical Association*, XV (1916–17), 388–404.

[132] See page 172.

"You know how Quetelet reduced the most apparently accidental carelessness to ever recurring facts, so that as long as the same conditions exist, the same 'accidents' will recur with absolutely unfailing regularity.

"You remember how Quetelet wrote—and Sir J. Herschel enforced the advice—'Put down what you expect from such and such legislation; after—years, see where it has given you what you expected, and where it has failed. But you change your laws and your administering of them so fast, and without inquiry after results past or present, that it is all experiments, see-saw, doctrinaire, a shuttlecock between two battledores.' "[133] In speaking of Florence Nightingale, Pearson says:[134] "Her statistics were more than a study, they were indeed her religion. For her, Quetelet was the hero as scientist, and the presentation copy of his *Physique Sociale*[135] is annotated by her on every page. Florence Nightingale believed—and in all the actions of her life acted upon that belief—that the administrator could only be successful if he were guided by statistical knowledge. The legislator—to say nothing of the politician—too often failed for want of this knowledge. Nay, she went further: she held that the universe—including human communities—was evolved in accordance with a divine plan; that it was man's business to endeavor to understand such a plan and guide his actions in sympathy with it. But to understand God's thoughts, she held we must study statistics, for these are the measure of his purpose. Thus the study of statistics was for her a religious duty."

Ebbinghaus. One of the earliest attempts to apply the law of error in a purely psychological study, as distinct from work in psycho-physical measurements, was made by Ebbinghaus[136] in 1885. His experimental investigation of memory[137] was a pioneer study in the application of the precise scientific methods of quantitative analysis in the field of the higher mental processes, and pointed the way for a long train of other investigators to follow. A comment, made by a reviewer[138] soon after the appearance of the book, is of some interest in view of the developments in professional education since that time: "His reticence tempts one into speculation as to the

[133] See Pearson, *Life of Francis Galton*, II, p. 418.

[134] *Ibid.*, p. 414.

[135] Now the property of the Galton Laboratory.

[136] Hermann Ebbinghaus (1850–1909), professor of philosophy at the University of Halle, was the author of many memoirs on psycho-physics, and the inventor of the completion test as a measure of intelligence.

[137] *Über das Gedächtnis*, Leipzig, 1885. A translation by H. A. Ruger is also available: *Memory, a Contribution to Experimental Psychology*, New York, 1913.

[138] Review by Jacobs in *Mind*, X (1885), 451–459.

future of the new branch of psychometry which he has opened up. May we hope to see the day when school registers will record that such and such a lad possesses 36 British Association units of memory-power or when we shall be able to calculate how long a mind of 17 'macaulays' will take to learn Book ii of *Paradise Lost.* If this be visionary, we may at least hope for much of interest and practical utility in the comparison of the varying powers of different minds which can now at last be laid down to scale."

From the point of view of psychology, the book is noteworthy because of Ebbinghaus's invention of the use of nonsense syllables as a medium for studying memory, because of his insight in conceiving the idea of using as a measure of memory the time saved in relearning, and because of the laws of memory which he stated. From the point of view of the history of statistics, the book is a landmark in the application of the theory of errors of observation to the examination of higher mental processes. The author's preface indicates clearly that the technique employed was an innovation. Ebbinghaus discussed the law of error at length in a non-technical fashion, drawing graphs, and finally referring his readers to text-books on probability for further information. He made extensive use of probable errors, and was punctilious in stating the probable error of every mean. He gave a short table showing the number of measures in each sample of one thousand which might be expected to fall by chance within certain limits, and compared the actual and theoretical number of cases within these limits. In this connection he quoted Lexis, whose book on mass phenomena[139] had appeared in 1877. In speaking of the closeness of the correspondence of his observations to the theoretical law of errors, he compared the exactness of his measures with the determination of the speed of nervous transmission as made by Helmholtz and Baxt, and also made reference to the probable errors of Joule's first determination[140] of the equivalent of heat.

Lexis. Among the many notable German writers of the last quarter of the nineteenth century who applied the theory of probability to the practical affairs of the world, brief mention must be made of Lexis.[141] He seems

[139] *Zur Theorie der Massenerscheinungen in der Menschlichen Gesellschaft*, Freiburg, 1877, 95 p.

[140] *Philosophical Magazine*, 1843.

[141] Wilhelm Lexis (1837–1914), studied mathematics and natural science at Bonne and social science at Paris. He was a professor of political economy at the universities of Strassburg (1872), Dorpat (1874), Freiburg (1876), Breslau (1884), and Göttingen (1887), a member of the Royal Statistical Society of London, of the Royal Academy of Sciences at Göttingen, of the Royal Economical Society of St. Petersburg, Vice-President of the International Statistical Association, and author of works on political economy and population statistics.

to have been the first[142] who successfully investigated the question of the extent to which a particular set of observations conforms to the law of error. In his work on the theory of mass phenomena,[143] he presented methods for answering this question empirically, and then stated the famous Lexian ratio[144] between the observed dispersion in a statistical series and the theoretical value of the standard deviation, $\sigma = \sqrt{npq}$. Johann von Kries, himself a writer of no mean merit in this field, thinks the work of Lexis just mentioned one of the most important in the whole field of the applications of the theory of probability,[145] because it was the first to enunciate clearly the important rules concerning the nature of the dispersion of a statistical series, and their bearing upon the normality of the distribution.

Galton. The use of the normal curve in assigning class marks and in scale making finds its inspiration in the writings of Francis Galton[146] and his associates of the English school. Indeed in many ways the daily regimen of the public school children of America exhibits traces of the thinking of that remarkable Englishman, while in still other ways we have not yet reached the goals he pointed out. Galton advocated, for example, that every child should have a complete psychological and physical examination, including, of course, anthropometric measurements, and should have a cumulative record card to follow him through school.

To exaggerate the influence of Francis Galton upon modern statistical method in education in the United States would be very difficult. It was

[142] For mention of the later work of Pearson and Charlier, see p. 81.

[143] *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, Freiburg, 1877.

[144] *Op. cit.*, p. 28. See Arne Fisher's *Mathematical Theory of Probabilities*, New York, 1926 (1st ed. 1915), pages 124–126.

[145] See Von Kries, *Die Principien der Wahrscheinlichkeitsrechnung*, Freiberg, 1886, p. 287.

[146] Sir Francis Galton (1822–1911), was a cousin of Charles Darwin. His two grandfathers were both Fellows of the Royal Society. His father had a strong interest in science and in statistical inquiry. Before entering Cambridge, Galton studied medicine in the Birmingham Hospital, and in London he studied anatomy and botany. Finding little opportunity at Cambridge for experimental and observational science, he devoted himself chiefly to mathematics. "Thus it came about that Galton had far more mathematics and physics than nine biologists out of ten, and more biology than nineteen mathematicians out of twenty, and more acquiantance with diseases and anomalies than forty-nine out of fifty biologists and mathematicians together." (Pearson.) After three years of university work his health broke and he resorted to travel down the Danube to the Black Sea, in Egypt, and unfrequented parts of Africa. Among the many topics upon which he studied and wrote were the following: The art of travel, mountain climbing, geography, laws of heredity, eugenics, history of twins, medicine and surgery, psychology,

his discovery and publication of the method of correlation which seems to have attracted Pearson to the field of statistical theory. Cattell was associated with Galton and helped him set up his Anthropometric Laboratory at South Kensington, and it has been noted[147] that Cattell's teaching at Columbia, together with the work of his celebrated students, was one of the most powerful forces in the inception of a statistical movement in American education. The influence of American anthropologists and biologists upon educational statistics has also been mentioned. Most of these were influenced even more strongly by the work of Galton and Pearson than by that of Quetelet. Educational literature is replete with coefficients of correlation, the direct invention of Galton. It is also to Galton that we owe the percentile system, including the quartile deviation and the median, for although Fechner had enunciated the latter and other continental writers had used similar measures, these have never been widely known in America. Nor were Quetelet and Ebbinghaus,[148] with their belief that mental functions could be studied statistically, generally familiar to American readers. It was Galton who enormously widened the range of topics susceptible of statistical treatment, by his firm belief that "Until the phenomena of any branch of knowledge have been submitted to measurement and number it cannot assume the dignity of a science."[149]

Galton's Use of the Normal Curve. Galton's own words most vividly describe his attitude toward the normal curve:

"My first serious interest in the Gaussian Law of Error [the normal curve]

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applications of Fechner's Law, systems of measuring and scoring traits of character, of ability and of merit, biometry, anthropometry, psychometry, composite portraiture, finger prints, and statistical methods. He possessed unusual powers of observation, a passion for collecting numerical data and astonishing cleverness in the invention of devices for recording and for analyzing such data, and an extraordinary versatility and a scientific curiosity that found every aspect of human life worthy of careful scrutiny and experimentation. Terman has estimated that in early childhood Galton's I.Q. must have been in the neighborhood of 200. See *American Journal of Psychology*, XXVIII (1917), 209–215.

For accounts of his life, see the following:

Galton, *Memories of My Life*, London, 1908.

Pearson, *Life, Letters and Labours of Francis Galton*, 2 vol., London, 1914 and 1924.

Pearson, *Francis Galton, 1822–1922, a Centenary Appreciation*, No. XI in the series *Questions of the Day and of the Fray*, 1922.

[147] See p. 152.

[148] Until the translation of his work on memory by Ruger in 1913.

[149] Galton's contributions to evolution and the laws of heredity fall outside the scope of this present work. For an appraisal of his significance for science in general, see Pearson's *Centenary Appreciation*.

was due to the inspiration of William Spottiswoode, who had used it long ago in a Geographical memoir for discussing the probability of the elevations of certain mountain chains being due to a common cause. He explained to me the far-reaching application of that extraordinarily beautiful law, which I fully apprehended. I had also the pleasure of making the acquaintance of Quetelet."[150]

"I need hardly remind the reader that the Law of Error upon which these Normal Values are based, was excogitated for the use of astronomers and others who are concerned with extreme accuracy of measurement, and without the slightest idea until the time of Quetelet that they might be applicable to human measures. But Errors, Differences, Deviations, Divergencies, Dispersions, and individual Variations, all spring from the same kind of causes. Objects that bear the same name, or can be described by the same phrase, are thereby acknowledged to have common points of resemblance, and to rank as members of the same species. . . . . This general statement is applicable to thousands of instances. The Law of Error finds a footing wherever the individual peculiarities are wholly due to the combined influence of a multitude of 'accidents' in the sense in which that word has already been defined. All persons conversant with statistics are aware that this supposition brings Variability within the grasp of the laws of Chance, with the result that the relative frequency of Deviations of different amounts admits of being calculated, when these amounts are measured in terms of any self-contained unit of variability, such as Q."[151]

"It has been objected to some of my former work, especially in Hereditary Genius, that I pushed the applications of the Law of Frequency of Error somewhat too far. I may have done so, rather by incautious phrases than in reality; but I am sure that, with the evidence now before me, the applicability of that law is more than justified within the reasonable limits asked for in the present book. I am satisfied to claim that the Normal Law is a fair average representation of the Observed Curves during nine-tenths of their course; that is for as much of them as lies between the grades[152] of 5° and 95°. In particular, the agreement of the Curve of Stature with the Normal Curve is very fair, and forms the mainstay of my inquiry into the laws of Natural Inheritance. It has already been said that mathematicians laboured at the Law of Error for one set of purposes, and we are entering into the fruits of their labours for another. Hence there is no ground for

[150] *Memories of My Life* (1908), p. 304.
[151] *Natural Inheritance* (1888), pp. 54, 55.
[152] The fifth and ninety-fifth percentiles.

surprise that their nomenclature is often cumbrous and out of place, when applied to problems in heredity."[153]

"But there is always room for legitimate doubt whether conclusions based on the strict properties of the ideal law of error would be sufficiently correct to be serviceable in actual cases of co-relation between variables that conform only approximately to that law. It is therefore exceedingly desirable to put the theoretical conclusions to frequent test, as has been done with these anthropometric data. The result is that anthropologists may now have much less hesitation than before, in availing themselves of the properties of the law of frequency of error."[154]

"I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the 'Law of Frequency of Error.' The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along."[155]

Two quotations from contemporaries of Galton's seem pertinent at this point. John Venn writes: "Mr. Galton . . . . has proposed to extend the same principles of calculation to mental phenomena, with a view to their more accurate determination. The objects to be gained by doing so belong rather to the inferential part of our subject, and will be better indicated further on; but they do not involve any distinct principle. . . . . That our mental qualities, if they could be submitted to accurate measurement, would be found to follow the usual Law of Error, may be assumed without much hesitation. The known extent of the correlation[156] of mental and bodily characteristics gives high probability to the supposition that what is supposed to prevail, at any rate approximately, amongst most bodily elements which have been submitted to measurement, will prevail among the mental elements. To what extent such measurements could be carried out practically is another matter. It does not seem to me that it could be done with much success. . . . . The doctrine, therefore, that mental

[153] *Natural Inheritance*, pp. 56, 57.

[154] "Co-relations and their Measurement," *Proceedings of the Royal Society*, XLV (1880), p. 144.

[155] *Natural Inheritance*, p. 86.

[156] This is not to be understood as correlation in the technical, statistical sense.

qualities follow the now familiar law of arrangement can scarcely be grounded on anything more than a strong analogy.[157]

Writing on Probability in the ninth edition of the *Encyclopaedia Britannica*, Sir John Herschel[158] says: "Men were surprised to hear that not only births, deaths, and marriages, but the decisions of tribunals, the results of popular elections, the influence of punishments in checking crime, the comparative values of medical remedies, the probable limits of error in numerical results in every department of physical inquiry, the detection of causes physical, social, and moral, nay even the weight of evidence and the validity of logical argument, might come to be surveyed with the lynx-eyed scrutiny of dispassionate analysis."

5. ABSCISSA UNITS OF THE NORMAL CURVE

Diversity of Terms. We shall now consider historically the origin of the following units of measure which have been used for the abscissa unit of the normal curve: "modulus," "probable error," "mean error," "middle error," "mean square error," "error of mean square," "mean deviation," "precision," "semi-interquartile range," "median deviation," "standard deviation," "fluctuation," "variance," "weight."

The Modulus. The first formulation of the law of error occurs in De Moivre's *Approximatio* (1733). In this treatise, however, no specific name is given either to the curve itself or to the units in which abscissas are measured. The latter are expressed as multiples of what would now be called the standard deviation, or in his symbolism, $\frac{1}{2}\sqrt{n}$. When five years later De Moivre incorporated the translation of this *Approximatio* into the second edition of *The Doctrine of Chances* (1738), he added two brief but significant passages, in one of which he says, " . . . . the Square-root of the number which denotes how many experiments have been, or are designed to be taken, ... will be as it were the Modulus by which we are to regulate our Estimation." Thus the word "modulus" which came later to mean $\sigma\sqrt{2}$, is here used for 2σ. The term "modulus" does not occur again in the treatise. No other writer appears to have used the term to mean 2σ.

In 1799 the French physicist Kramp published[159] a rather extensive

[157] *The Logic of Chance*, 3rd ed. 1888, pp. 48, 49.

[158] Sir John Frederick William Herschel (1792–1871) only child of the astronomer, Sir William Herschel, studied mathematics, law, chemistry, physics, and finally devoted himself to astronomy. Two of his sons also wrote on astronomy.

[159] *Analyse des Réfractions Astronomiques et Terrestres* (Strasburg, 1799).

table of the integral $\int e^{-tt}\,dt$. Here $t = \dfrac{x}{\sigma\sqrt{2}}$ in our present notation. These tables were repeatedly copied and extended and formed the basis for most of the numerical work in probability up to 1903, when Sheppard published his well-known tables[160] with the argument $\dfrac{x}{\sigma}$.

The value of $\dfrac{1}{\sigma\sqrt{2}}$ was called by Gauss the "measure of precision of the observations,[161] and was represented by the letter h. The equation for the curve he wrote $\phi\Delta = \dfrac{h}{\sqrt{\pi}}\, e^{-hh\Delta\Delta}$. It is interesting to note that both t and h are still often used in theoretical treatises on probability. Encke (1832) called h "the weight" (*das Gewicht*). Airy[162] used the term "modulus" for $\sigma\sqrt{2}$, and the term has persisted to the present time.

The Probable Error. The term probable error originated among the German mathematical astronomers who wrote near the beginning of the 19th century. The early use of the term is in certain memoirs dealing with astronomy, geodesy, or artillery fire, where the writer is attempting to make the best possible determination of the true position of a point from a series of observations all of which involve an element of error. A deviation from the true position of the point, or more commonly from the mean of the observations, of such a magnitude that, if the number of observations be indefinitely increased, one half of the errors may be expected to be numerically greater and one half numerically less than this value, is then termed the "probable error." When the frequencies of the various errors are plotted, the result is quite naturally spoken of as the "curve of facility of error," or "curve of error," and the formula describing it as the "law of facility of error," "law of error," and "error function." About equally frequent, but directing attention to the theoretical background of the law, we find "law of probability," "law of normal probability," "probability curve," "curve of normal probability," and finally "normal curve."

The first recognition by any writer of a deviation such that between its positive and negative values one-half of the cases may be expected to fall

[160] W. F. Sheppard, "New Tables of the Probability Integral," *Biometrika*, II. These were later reprinted in Pearson's *Tables for Statisticians and Biometricians*.

[161] "Ceterum constans h tamquam mensura praecisionis observationem considerari poterit," *Theoria Motus* (1809), §178.

[162] *Theory of Errors of Observations*, 2d ed. (1875), 15.

by chance, is found in the second[163] of the two passages which De Moivre added to his *Approximatio* in 1738. He gives this value no name, and does not use it as a unit of measure, but he finds it to be approximately equal to $\frac{\sigma}{2}\sqrt{2}$, a result differing by less than 5 per cent from the correct value.

The term *probable error* appears to be due to Bessel, who used the expression *der wahrscheinliche Fehler* in 1815.[164] The term was immediately adopted by Gauss. In a paper on the determination of the accuracy of observations[165] which he wrote in 1816, Gauss defined the probable error,[166] represented it by the letter r, and gave three methods of computing it. One of these is somewhat similar to the computation of Galton's Q. He also stated the most probable value for r to be $0.6744897\sqrt{\frac{\alpha\alpha+\beta\beta+\gamma\gamma+\text{etc.}}{m}}$ [that is, p.e. $= 0.6744897\sigma$] and the most probable value of h to be $H=\sqrt{\frac{m}{\alpha\alpha+\beta\beta+\gamma\gamma+\text{etc.}}}$ [that is, the best value for the ordinate of the probability curve is $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\Delta^2}{2\sigma^2}}$]. Gauss also gave here a brief table in which deviations were expressed as multiples both of the modulus and of the probable error.

Bessel published a paper in 1816 in which he defined the probable error and gave two formulas which are the equivalents of our formulas p.e. $= 0.6745\sigma$ and p.e. $= 0.8453$ mean deviation.

Plana used *der wahrscheinliche Fehler*, without any explanation of its

[163] " . . . . we may naturally be led to inquire, what are the Bounds, within which the proportion of Equality is contained; I answer that these bounds will be set at such a distance from the middle Term, as will be expressed by $\frac{1}{4}\sqrt{2n}$ very near . . . . so that it is an equal Chance nearly, or rather something more, that in 3600 Experiments in each of which an Event may as well happen as fail, the Excess of the happenings or failings above 1800 times will be no more than about 21."

[164] The following passage occurs in Bessel's "Ueber den Ort des Polarsterns" (*Berliner Astronomisches Jahrbuch für* 1818): "Die Grunde dieser Schätzung des wahrscheinlichen Fehlers, beruhen auf der von Gauss gegebenen Entwickelung der Wahrscheinlichkeit, einen Fehler von gegebener Grösse zu begehen; ihre Mitteilung muss ich bis auf eine andere Gelegenheit versparen." p. 234.

[165] "Bestimmung der Genauigkeit der Beobachtungen," *Zeitschrift für Astronomie und verwandte Wissenschaften*, I, 185–196.

[166] "Die Wahrscheinlichkeit, dasz der Fehler nicht unter ρ/h ist, ist also $= \frac{1}{2}$, oder der Wahrscheinlichkeit des Gegentheils gleich: wir wollen diese Grösze den *wahrscheinlichen Fehler* nennen, und mit r bezeichnen," p. 186.

meaning, in a memoir written in 1818,[167] as though the phrase *probable error* was already well understood by German astronomers.

By 1832, when Encke wrote "Über die Methode der kleinsten Quadrate,"[168] the terminology appears to have become standardized in Germany, and he remarks that the quantity r (which is the probable error) is called by the German astronomers *der wahrscheinliche Fehler* while the French mathematicians are accustomed to give it the name *mean error* (*l'erreur moyenne*), a term which must be carefully distinguished from the middle error (*der mittlere Fehler* = our standard deviation) of the Germans. It is of some interest to compare Encke's terminology with the symbolism most commonly used today.

MODERN ENGLISH FORMS		FORMS USED BY ENCKE	
Symbol	Term	Symbol	Term
σ	standard deviation standard error	ε_2	die mittlere Abweichung der mittlere Fehler
p.e.	probable error	$r = \frac{\rho}{h}$	der wahrscheinliche Fehler
$\frac{1}{\sigma\sqrt{2}}$		h	das Gewicht

Gauss used *error probabilis* in some of his Latin works, as *Theoria Combinationis Observationum.* Quetelet called the measure *l'erreur probable* and used the letter ω to represent it and Q to represent our standard deviation, so that he wrote $\omega = 0.67450\ Q$.

Confusion in English Terminology. Although the Germans, thanks to the dominating genius of Gauss, and the French, probably because of the leadership of Laplace, were able to achieve a degree of unanimity in their terminology at an early date, the English were less fortunate. Until the time of Pearson, England produced no single outstanding creative writer on the subject of probabilities or the mathematical theory of statistics.

[167] "Allgemeine Formeln um nach der Methode der kleinsten Quadrate die Verbesserungen von 6 Elementen zu berechnen und zugleich das jeder derselben zukommenden Gewicht zu bestimmen," *Zeitschrift für Astronomie*, 1818, 249–264.

[168] *Berliner Astronomisches Jahrbuch für* 1834.

Instead there were a number of very able scholars, most of whom added one or more terms to the not inconsiderable list already in use.

Strange as it may seem, the first of these terms to approach standardization, was the somewhat misleading expression *probable error*, taken over by direct translation from the German. This had apparently reached general acceptance in 1836 when De Morgan wrote his "Theory of Probabilities" in the *Encyclopaedia Metropolitana*. That the term is inadequate to its present-day duties is generally recognized, but its historical sanctions are well established. Galton pays his respects to it in the following fashion: "It is astonishing that mathematicians, who are the most precise and perspicacious of men, have not long since revolted against this cumbrous, slipshod, and misleading phrase. They really mean what I should call the Mid-Error, but their phrase is too firmly established for me to uproot it. I shall however always write the Probable when used in this sense, in the form of 'Prob;' thus 'Prob. Error,' as a continual protest against its illegitimate use, and as some safeguard against its misrepresentation. Moreover the term Probable Error is absurd when applied to the subjects now in hand, such as Stature, Eye-color, Artistic Faculty, or Disease. I shall therefore usually speak of Prob. Deviation."[169] Venn makes the similarly pertinent comment: "When we perform an operation ourselves with a clear consciousness of what we are aiming at, we may quite correctly speak of every deviation from this as being an error; but when Nature presents us with a group of objects of every kind, it is using a rather bold metaphor to speak in this case also of a law of error, as if she had been aiming at something all the time, and had like the rest of us missed her mark more or less in every instance."[170]

Other Measures of Dispersion. While the enumeration which follows is far from exhaustive, it will serve to illustrate the multiplicity of terms which have been applied to measures of dispersion. De Morgan, in his *Essay on Probabilities* (1838), a treatise written from the popular point of view, speaks of "*the* square root," by which he means the quantity $\sqrt{\frac{8nab}{a+b}}$ in the formula $t = \frac{2l+1}{\sqrt{\frac{8nab}{a+b}}}$, a formula equivalent to $t = \frac{2x}{\sigma\sqrt{8}}$. The expression "*the* square root" appears to have received no sanction from any

[169] *Natural Inheritance* (1889), p. 58.
[170] *The Logic of Chance* (1888), p. 42.

other writer. He uses the term "average balance" for $\frac{\Sigma \text{ deviations}}{N}$, "average error" for $\frac{\Sigma \text{ (numerical values of the deviations)}}{N}$, "mean risk of error"[171] for $\frac{\Sigma \text{ (deviations of one sign)}}{N}$, "probable error," and "standard law of facility of error," but he has no term for the standard deviation itself. He uses the letter e to represent the probable error, and suggests[172] the term *critical error* as an improvement on *probable error.*

Airy[173] used a great variety of terms, as if uncertain which might eventually prove most valuable, as for instance:

c	= Modulus (*i.e.*, $c = \sigma\sqrt{2}$)	page 17.
m.e.	= Mean Error ($= c/\pi$)	page 20.
$\frac{c^2}{2}$	= Mean Square ($= \sigma^2$)	page 21.
e.m.s.	= Error of Mean Square ($= \sigma$)	page 21.
p.e.	= Probable Error	page 21.

Lexis spoke of the "dispersion," Edgeworth of the "fluctuation," and R. A. Fisher of the "variance." A complete list of the symbols which have been used for the probable error would be a very long one. Porter[174] used d for the probable deviation, and Wissler (1901) used p.

The Standard Deviation. The term standard deviation was proposed by Pearson[175] and is now used by almost all English writers. As originally defined by Pearson, this is the square root of the mean of the squares of deviations *taken from the mean of the distribution*, and is not to be used when

[171] " . . . . the mean risk of error, to which a probable approximation might be made by taking the average amount of a large number of errors. The name of the *probable error* has sometimes been given to this result, but incorrectly. If by probable error be meant the most probable of all errors, the answer should be nothing, for the correct truth is individually more probable than any given error." Reprint of article on probability in *Encyclopaedia Metropolitana* as it appears in *Mathematical Papers of De Morgan*, p. 445.

[172] "On the Theory of Errors of Observation," *Cambridge Philosophical Transactions*, X, (1864), 409–427.

[173] Sir George Biddel Airy (1801–1892), Astronomer Royal of England, *Theory of Errors of Observations* (1861), 2nd ed. revised 1875. Page references given here are to the second edition.

[174] "On the Application to Individual School Children of the Mean Values Derived from Anthropological Measurements by the Generalizing Method," *Papers on Anthropometry* (Boston, 1894).

[175] "Contributions to the Mathematical Theory of Evolution.—I. On the Dissection

deviations are measured from any other reference point. Pearson uses the term "root-mean-square" for a similar measure when the deviations are taken around any origin other than the mean. Therefore a "standard deviation from the median" cannot be reconciled with this definition. Unless this distinction is preserved it is impossible to insure that all computations of the standard deviation from the same data shall yield the same results.[176]

The Mean Deviation. Pearson seems to be the originator of the term "mean error" or "mean deviation" as applied to the arithmetic mean of the numerical values of the deviations. Quetelet had, however used a similar expression in his *Lettres*, where he says on page 398, that the mean error is the sum of the products of each individual error by its probability.[177]

Probability Expressed as a Function of the Deviation. When the distribution of a variate conforms to the normal law of error, and the number of cases is very large, then it is possible to determine the probability of the chance occurrence of a deviation corresponding to any given value of $\frac{x}{\sigma}$, $\frac{x}{\text{p.e.}}$, $\frac{x}{\sigma\sqrt{2}}$, or of x divided by any desired multiple of a measure of the variability. This we have seen to be the usual procedure since the time of De Moivre. Gauss expressed this probability as a function both of $\frac{x}{\sigma\sqrt{2}}$ and of $\frac{x}{\text{p.e.}}$ (to use modern symbolism). Quetelet and Galton both showed a preference for the ratio $\frac{x}{\text{p.e.}}$, although they used the other also. It has

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of Asymmetrical Frequency-curves," *Philosophical Transactions*, A, 1894. On page 80 he writes, "Let the equation to the probability curve be

$$y' = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-x^2/2\sigma^2}.$$

Then σ will be termed its *standard-deviation*, (error of mean square);" and in a footnote on page 88, "I have always found it more convenient to work with the standard deviation than with the probable error or the modulus, in terms of which the error-function is usually tabulated."

176 See W. C. Eells, "A Plea for a Standard Definition of the Standard Deviation," *Journal of Educational Research*, XIII (1926), 45–52.

177 "On fait également usage de *l'erreur moyenne*, qui est la somme des produits de chaque erreur particuliere par sa probabilité."

also been a fairly common practice to assume that when $x = 3\sigma$, it is "practically certain" that the deviation has not arisen solely as the result of chance errors in the sampling, but must have been due to some significant cause. Any numerical definition of "practical certainty" must be more or less arbitrary, inasmuch as certainty is a subjective matter, and odds which seem practically certain to one man in a given situation might well be unconvincing to another man in another situation. However $x = 3\sigma$ is a very convenient limit and most people would be convinced by odds of 369 to 1. Recently McCall and McGaughy have each given a name to a particular fraction of this type. McGaughy applies the term "critical ratio" to the value $\frac{x}{\text{p.e.}} = 3$, where x represents a difference. McCall uses the term "experimental coefficient" for the ratio $\frac{x}{2.78\sigma}$.[178]

The Median Deviation. Probably the first person to use a measure of dispersion similar to what we now call the *median deviation* was Gauss. In 1816[179] he described an alternative method of computing the probable error which he said was simpler though less exact. His method was as follows: The m errors of observation are to be arranged in order of their size (without regard to signs). If the number is odd, the middle one, and if the number is even the mean of the two middlemost errors, is the measure sought, and is to be called M. He stated that when the number of observations is very large, M tends to coincide with the probable error. He also measured the degree of precision of M,[180] and stated that 249 observations

[178] There seems to be some temptation to an investigator to report only the size of the experimental coefficient or the critical ratio, and not to give the exact probability that a given difference is due to chance. If such extreme simplification encourages the investigator to discard valuable information already at hand, and to make a categorical rather than a quantitative statement concerning the significance of the difference, it is not an asset. Another possible source of confusion is the inference which might be drawn that an experimental coefficient—or a critical ratio—equal to .70 is "70% as significant" as an experimental coefficient equal to one. Some care needs to be taken by investigators to guard against these misunderstandings.

[179] "Bestimmung der Genauigkeit der Beobachtungen," *Zeitschrift für Astronomie*, I, 187–197.

[180] He says that the probable limits of M are $r\left(1 \mp e^{-\rho\rho}\sqrt{\frac{\pi}{8m}}\right)$, and the probable limits of r are $M\left(1 \mp e^{-\rho\rho}\sqrt{\frac{\pi}{8m}}\right)$, or, in numbers, $M\left(1 \mp \frac{0.7520974}{\sqrt{m}}\right)$.

would be needed to secure as great accuracy in M as would be obtained in computing $\frac{\Sigma |x|}{N}$ from 100 observations.

A Generalized Measure of Dispersion. There is logically no reason why $\sqrt{\frac{\Sigma x^2}{m}}$ should be the only measure of the type $\sqrt[n]{\frac{\Sigma x^n}{m}}$ to be used as a measure of variability. Gauss generalized this to include $\varepsilon_n = \sqrt[n]{\frac{\Sigma x}{m}}$ where the sign of x is to be considered as always positive. He had the values of ε_n for the normal curve from $n = 0$ to $n = 7$, and gave values for the probable errors of all these. Thus his ε_1 would correspond to our *mean deviation.* When n is even, this ε_n also corresponds to the nth root of Pearson's nth moment coefficient. Obviously the moments also form a system of measures of deviation. Fechner[181] carried this idea a step further, and defined a set of means (*Potenz-mittelwerthe*) such that each would represent an origin around which one of these various ε_n's would be a minimum. It was in this fashion that he arrived at the median (*Centralwerth*), inquiring around what point of reference the sum of the absolute values of the deviations would be a minimum.

The Quartile Deviation. The use of the semi-interquartile range as a measure of dispersion is a direct outgrowth of Galton's use of percentiles, and has been discussed under that heading.

6. TABLES OF THE NORMAL PROBABILITY FUNCTION

De Moivre. The first enunciation[182] of the formula[183] for the normal curve (1733) was not accompanied by any tables giving values of the areas

[181] "Über den Ausgangswerth der kleinsten Abweichungssumme, dessen Bestimmung, Verwendung, und Verallgemeinerung," *Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, Mathematisch-Physische-Classe,* XI (1878).

[182] See page 13.

[183] The formula for an ordinate of the normal curve is now usually written $y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, or for the normal probability curve in which the area is taken as unity, as $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$. In earlier treatises this was usually written as $y = \frac{1}{\sqrt{\pi}} e^{-tt}$, t being equal to $\frac{x}{\sigma\sqrt{2}}$. Obviously the area under any portion of the curve is given by the value of $\int \frac{1}{\sqrt{\pi}} e^{-t^2}\, dt$ within appropriate limits.

or the ordinates of that curve. However, by the aid of mechanical quadratures, De Moivre computed the area contained between the ordinate at the mean and the ordinates at the points which we would now designate as $x=\sigma$, $x=2\sigma$, and $x=\frac{3}{2}\sigma$. Five years later, when he published an English version of the *Approximatio*, he had determined a value for x such that the area between the ordinates at $+x$ and at $-x$ should be one-half the total area under the curve, that is, $x = p.e.$

First Tables of the Probability Integral.[184] Laplace had suggested in 1783 that a tabulation of the values of the integral of e^{-t^2} would be useful.[185] The first form of those tables of the areas under the normal curve which are today found in almost every text on statistics, was printed in 1799 in a book on refractions by the French physicist Kramp.[186]

This book was not a treatise on probability, but was a study of the measurement of refractions near the horizon. It contains three important tables, as follows:

Table I. Values of $\int e^{-tt}\,dt$ to the argument t from $t = 0.00$ to $t = 3.00$ by intervals of 0.01, 8 to 11 decimals, three columns of differences. This

[184] For other accounts of these tables, see J. W. L. Glaisher, article on "Tables" in the eleventh edition of the *Encyclopaedia Britannica*, and also "List of Tables of $\int e^{-t^2}\,dt$ and Connected Integrals and Functions," *Messenger of Mathematics*, XXXVIII (1908), 117–128.

[185] See page 20.

[186] *Analyse des Réfractions Astronomiques et Terrestres, par le citoyen Kramp*, Strasburg and Leipzig, 1799. The date is often given as 1798. Merriman says 1799. Glaisher says 1798. The copy which I read was marked "Imprimé a Strasbourg . . . . l'an de la Republique VII et se trouve a Leipsic . . . . MDCCLXXXVIIII." The seventh year of the Republic began Sept. 22, 1798. Kramp was a professor of chemistry and experimental physics in the central school of the department of the Ruhr. Besides the first tables of the normal curve, the book contains other matters of great interest to workers in the mathematical theory of statistics. It introduces the term *facultés* for products of factors of the form $a\,(a + r)\,(a + 2r)\,(a + 3r)\ \ldots\ (a + nr)$, a class of functions which the author says he thinks merits a name. Here too is the first use of the notation of the gamma function. "Je me sers pour la première fois de la notation $\Gamma\frac{r}{a}$, $\Gamma\frac{r}{n + mr}$. Il me resta à expliquer à quoi sorte ici la lettre grecque *gamma*. Je déclare donc que je désignarai géneralement par Γy la serie

$$\beta y + \tfrac{1}{3}\,\delta y^3 + \tfrac{1}{5}\,\zeta y^5 + \tfrac{1}{7}\,\theta y^7 + \tfrac{1}{9}\,\kappa y^9 + \tfrac{1}{11}\,\mu y^{11} + etc."$$

In another place he gives the values of these coefficients and the equations by which they are connected. He also states that $\Gamma 0 = 0$ and that $\Gamma 1 = 1 - \frac{1}{2}\log 2\,\pi$.

corresponds to our tables of the areas under the normal curve, except that the argument used by Kramp was $\frac{x}{\sigma\sqrt{2}}$ while the tables used today are usually prepared for the argument $\frac{x}{\sigma}$.

Table II. Logarithms of $\int e^{-tt}\,dt$ from $t = 0$ to $t = 3.0$, by intervals of 0.01, seven decimals, two columns of differences.

Table III. Logarithms of the product $e^{tt} \int e^{-tt}\,dt$ from $t = 0.00$ to $t = 3.00$ by intervals of 0.01, seven decimals, two columns of differences.

These tables facilitated the development of a practical theory of errors of observation based on the curve of error. They had a great influence throughout Europe and were frequently copied and extended. Nearly all the writers on probability of last century either copied these tables, or copied from someone else who had copied them.

Use of Error Function by Astronomers. Kramp had shown that the error function, $\int_t^\infty e^{-tt}\,dt$ was of fundamental importance in problems relative to astronomical and terrestrial refraction.[187] In the fourth volume of Laplace's work on celestial mechanics,[188] part of the tenth book is devoted to refractions, and the error function is given a prominent place. Fourier made use of the same function in his work on the conduction of heat. Gauss, in his study of the determination of the accuracy of observations,[189] presented a brief table of the functions of the probability integral. The argument used for this table is the area known as α in Sheppard's Tables, that is, the area between two ordinates equally distant from the mean. There are nine entries in this column of areas. For each of the nine areas, there is given the corresponding value of the abscissa, expressed first as a multiple of the modulus $\sigma\sqrt{2}$ and then as a multiple of the probable

[187] For further elaboration of this point, see Glaisher, "On a Class of Definite Integrals," *Philosophical Magazine*, 4th Series, XLII (1871), 421–436.

The earliest writers wrote the exponent as $-tt$ instead of $-t^2$. Obviously when the limits are taken from t to infinity, the value of the integral gives the area of the shorter tail of the distribution. When the limits are taken from 0 to t, the integral gives that portion of the area under the curve included between the ordinate at the mean and the ordinate at the point t.

[188] *Mecanique Celeste*, Vols. I and II (1799), III (1802), IV (1805), V (1825). An analysis of the content is found in the *English Cyclopaedia*.

[189] "Bestimmung der Genauigkeit der Beobachtungen," *Zeitschrift für Astronomie*, I (1816), 187–197.

error. The first two rows of the table when translated read thus: Let θ be the value of the integral $\int \frac{2e^{-tt}}{\sqrt{\pi}}\,dt$ from $t = 0$ on. Then we have:

$$0.5000000 = \theta \quad 0.4760363 = \theta\,\rho$$
$$0.6000000 = \theta \quad 0.5951161 = \theta\,\rho \quad 1.247790$$

This last line may be interpreted to mean that the central strip which includes 60 per cent of the area under the curve lies between the ordinates at the points $\pm 0.5951161\ \sigma\sqrt{2}$, which are the same as the points ± 1.247790 p.e. The complete table is reproduced on page 185 of *The Calculus of Observations* by Whittaker and Robinson.[190] Gauss has two innovations here. One is the use of the probable error as a unit of measure for abscissas. The other is giving not the area of the smaller tail beyond the point t, as Kramp had done, but the area of a central section symmetrical with reference to the maximum ordinate. In general, physicists have made more use of tables organized as were Kramp's, while educators, psychologists, and scientists concerned primarily with errors of observation have usually found it more convenient to read the area of the central section directly from the table.

Kramp's tables did not go beyond $t = 3$, which in our notation would be $x = 3\sigma\sqrt{2}$, but the range was extended by Bessel[191] to $t = 10$. In 1826 Legendre published a table of the values of $2\int_t^\infty e^{-t^2}\,dt$ carried out to ten decimal places and extending as far as $t = .50$, in his work on elliptical functions. At the end of the article on least squares[192] in the *Berlin Astronomical Yearbook for* 1834, Encke gave two tables of the area α under the probability curve, the first developed for the argument $\frac{x}{\sigma\sqrt{2}}$ and the second for $\frac{x}{\text{p.e}}$. The first he says he developed from Bessel's table in the *Fundamenta*. The other may have been suggested by the short table given by Gauss in 1816, but the calculations were probably original. This was probably the first long table to use the probable error as an argument.

[190] London, 1924.

[191] *Fundamenta Astronomiae pro MDCCLV Deducta ex Observationibus James Bradley in Specula Astronomica Grenovincensi per 1750–1762 institutis*, Regiomonti, 1818, pp. 36, 37. Bessel computed values of the function $e^{t^2}\int_t^\infty e^{-t^2}\,dt$ for the argument $\log_{10} t$, from 0 to 1, to seven decimal places, with second order differences.

[192] "Über die Methode der kleinsten Quadrate," *Berliner Astronomisches Jahrbuch für 1834*, published in 1832.

The first of these tables was carried to seven decimal places, with second differences, and proceeded from $t = 0$ to $t = 2.00$ at intervals of 0.01. The second was carried to five decimals with first differences only, and proceeded from $t = 0$ to $t = 3.40$ by intervals of 0.01 and from $t = 3.40$ to $t = 5.00$ by intervals of 0.1.

Both of Encke's tables were reprinted by De Morgan four years later in an article on probability in the *Encyclopaedia Metropolitana*, as well as two other tables adapted from those published by Kramp in his treatise on refractions. From this time on the tables were well known to the English speaking world. An anonymous article on the "Mean" in the *Penny Encyclopaedia* (1839) gave a brief table with instructions for its use in finding "the probability that truth lies within a certain degree of nearness to the average." The table has as its argument $\frac{100\,x}{\sigma\sqrt{2}}$, and the instructions are palpably in error, or misprinted. An article by Galloway[193] printed in both the seventh and the eighth editions of the *Encyclopaedia Britannica* (1842, 1859) published one set of these tables without stating the source from which they were taken.[194] Airy published a table of values of the ordinates of the normal probability curve on page 16 of his *Theory of Errors of Observations*,[195] and on page 22 a table of the areas under the curve, which he said was taken from Kramp's *Refractions* and from the article by De Morgan in the *Encyclopaedia Metropolitana.* Airy also drew a picture of one half of the normal curve, indicating the points whose abscissas represented the probable error, mean error, error of mean square, and modulus.

In the latter half of the century, several very elaborate tables of the probability integral with $\sigma\sqrt{2}$ as unit were published by Glaisher (1871),[196]

[193] Thomas Galloway (1796–1851), Scottish mathematician and actuary.

[194] The practice of reproducing tables without reference to their source is not confined to this era. A great many modern texts in statistics contain some version, often an abridged version, of Sheppard's Tables, or of some of the other tables in Pearson's *Tables for Statisticians and Biometricians*, but not all of the authors of these texts have published the name of the original computer or made an acknowledgement of the stupendous labor he performed that the labors of others might be lightened.

[195] 2nd ed. 1875.

[196] J. W. L. Glaisher gave a table of $\int_t^\infty e^{-t^2}\,dt$ from $t = 3$ to $t = 4.50$ at intervals of .01, the results being carried 11, 13, and 14 places, in his paper "On a Class of definite Integrals, Part II," *Philosophical Magazine*, Series 4, XLII (1871).

Von Oppolzer (1882),[197] Radau (1885),[198] Markov (1888),[199] Kämpfe (1893),[200] and Burgess (1898)[201] but these were not widely used by persons applying the theory of probability to social problems. These more often followed in the footsteps of Quetelet, Gauss, and Ebbinghaus in the use of the probable error as unit, until Sheppard computed his tables with the standard deviation as unit.

Sheppard's Tables. The first tables to express the normal probability integral in terms of the standard deviation were computed by W. F. Sheppard[202] and published in 1902 in the second volume of *Biometrika*, with an editorial promise of a further provision of "numerical tables tending to reduce the labour of statistical arithmetic." Before this time, as we have seen, either the modulus, $\sigma\sqrt{2}$, or the probable error had been used as a unit for all such tables. The former is satisfactory for theoretical work but

[197] Theodor Ritter von Oppolzer, gave a table of $\int_0^t e^{-t^2}dt$ from $t = 0$ to $t = 4.52$ at intervals of .01, results being carried to ten places, and a table of $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$ from $t = 0$ to $t = 2$ at intervals of .01, results carried to five places, in his *Lehrbuch zur Bahnbestimmung der Kometen und Planeten, Leipzig*, 1st ed. 1880 (unverified), 2nd ed. 1882. The calculations were original.

[198] M. R. Radau gave a long table of $\log_{10} e^{t^2}\int_t^\infty e^{-t^2}dt$ from $t = -0.120$ to $t = 1.000$ at intervals of .001, carried to seven places, with differences, in "Tables de l'integrale $\psi(z) = e^{z^2}\int_z^\infty e^{-t^2}dt$," *Annales de l'Observatoire de Paris*, XVIII (1885), B. 1 à B. 25.

[199] Andrei Andreevich Markov published his *Tables des Valeurs de l'Integrale* $\int_x^\infty e^{-t^2}dt$ at St. Petersburg in 1888. These ranged from $t = 0$ to $t = 3$ by intervals of .001 and from 3 to 4.80 by intervals of .01, with third order differences, carried to eleven places. He also gave values of $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$ from 0 to 2.499 at intervals of .001 and on to 3.79 at intervals of .01.

[200] Bruno Kämpfe published a table of the integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$ in Wundt's *Philosophische Studien* IX (1893), 147–150.

[201] James Burgess published a very extensive set of tables which were the result of a new calculation, and which extended from 0 to 1.250 at intervals of .001 carried to nine decimals, from 1.00 to 3 carried to fifteen places, and from 3 to 6 at intervals of .1 carried to fifteen places, in the *Transactions of the Royal Society of Edinburgh*, XXXIX, Part II (1897–1898), 257–321.

[202] "New Tables of the Probability Integral," *Biometrika* II (1902–03), 174–190.

calls for a troublesome amount of arithmetic in computations with actual data; the latter has neither theoretical neatness nor economy of computation to recommend it. Sheppard's Tables have now completely supplanted the earlier ones so far as the social and biological sciences are concerned, and form the basis of all tables of the areas and ordinates of the normal curve published in recent texts. They have saved incalculable time and labor for every worker in educational statistics, even for those who—using some abridged form of these tables, taken from a text whose author neglected to acknowledge the source—have never heard of the computer.

The first of these tables is an extensive one, giving the areas and ordinates in terms of the abscissas. The second is the inverse of the first one, giving abscissa and ordinate in terms of the area of the central section, α. The argument proceeds from $\alpha = 0$ to $\alpha = .80$ by intervals of .01, with third order differences. Kelley and Wood[203] later extended this table to give values of the abscissa and ordinate, the proportion of area in each tail of the distribution, the product of the proportions in the two tails, and the ratio of the measure of the ordinate to that of the area of each tail, the argument being the area between the ordinate at the mean and the ordinate at the point x. The argument is tabled for values from $I = .000$ to $I = .499$ by intervals of .001, no differences, six decimal places only.

The first of Sheppard's Tables was extended by Miss Julia Bell[204] of the Biometric Laboratory Staff, to give the logarithms of areas and ordinates for deviations as large as $x = 500\sigma$.

Somewhat later Sheppard computed still another table[205] of great practical use to the statistician. This table gives the deviation from the mean of the normal curve, in terms of the standard deviation, corresponding to the ordinates which divide the area into 1000 equal parts, or the "permilles" of frequency[206] and was an expansion of a shorter table published by Galton in *Natural Inheritance* (1899), under the title "Per Centiles."

All three of Sheppard's Tables and Miss Bell's extension are available in that invaluable set of *Tables for Statisticians and Biometricians*[207] issued by the Biometric Laboratory of University College, University of London.

[203] Truman L. Kelley and Ben D. Wood, assisted by Mrs. Lura Kelley. The table is published in Kelley's *Statistical Method* (New York, 1924), p. 373–384.

[204] "Extension of the Tables of the Probability Integral F = $\frac{1}{2}(1 - \alpha)$," originally published in *Drapers' Research Memoirs, Biometric Series* VIII, p. 27.

[205] "Tables of Deviates of the Normal Curve for each Permille of Frequency," published by Galton in an article on "Grades and Deviates," *Biometrika* V (1906–07), 405.

[206] McCall's table for use in constructing T-Scales, shown in *How to Experiment in Education* (1923), page 101, is an abridged form of this table.

[207] 1st ed. 1914, 2nd ed. 1924.

7. PROBABLE ERRORS OF FREQUENCY CONSTANTS[208]

Work Done before 1890. The mathematical astronomers of last century knew the formulas for the probable errors of such statistical measures as do not involve correlation. They knew the probable error of a mean Gauss, 1809), of measures of variability of the normal curve (Gauss, 1809, Encke, 1832), of the kth root of the kth moment[209] (Encke, 1832), of the sum of independent variables (Encke, 1832), of the weighted sum of independent variables (Encke, 1832). Airy recognized and clearly stated that the formula which he gave for the probable error of $X + Y$ or of $X - Y$ could apply only when X and Y were independent, and in his discussion of "entangled measures" he indicated a realization that a satisfactory formula would have to take some account of possible interrelations between the two variables.

Some of the present day methods of computing probable errors are far from new. Encke published the "gross score" formula[210] for the standard deviation in 1832. T. W. Wright wrote in 1882[211] that two methods of computation of the probable error were then in use, the one in which the probable error was determined by means of the standard deviation[212] he credited to Bessel, the one in which the probable error was determined from the mean error he ascribed to Peters. Wright suggested further some rough rules for estimating the probable error from the range and the number of cases in the distribution.

Until Francis Galton gave to the world the coefficient of correlation, a satisfactory theory of the effects of sampling upon statistical constants and an adequate discussion of probable errors was manifestly inconceivable. This, like almost all other aspects of statistical theory, made great advances during the closing years of the 19th century and the first years of the 20th.

Pearson, Filon, and Sheppard. The fourth paper in Pearson's epoch-making series of Mathematical Contributions to the Theory of Evolution

[208] There is a certain anomaly in speaking of the *variability* of frequency *constants*. R. A. Fisher has used the expression *a statistic* as a class name for such frequency constants as the mean, median, standard deviation, and the like, but the term has not come into general use.

[209] See qualification on page 72.

[210] This fact throws an interesting light upon recent discussions as to which of several men now living invented the method.

[211] "On the Computation of Probable Errors," *The Analyst*, IX (Des Moines, Iowa, 1882), 74–78.

[212] The term *standard deviation* was not in use at this time, having been introduced by Pearson in 1893.

was devoted to the probable errors of frequency constants, and the influence of random selection on variation and correlation. Here Pearson and Filon (1898) established the mathematical theory and the general method of procedure in so fundamental a fashion that later work has but built on this foundation. The writers treated in turn the probable errors and the error correlations of the frequency constants of a system of two normally distributed variables, of three, and of four normally distributed variables, and of the constants of skew distributions. The particular results reached in this paper have been discussed more fully in the chapter on "Moments." Still more comprehensive was the work done by Pearson on the same subject in a series of papers[213] in *Biometrika,* after the invention of the p-notation for product moments.

Later Work by Many Writers. After these fundamental treatises had established the general theory and had shown the method of attack, there appeared a large number of other papers dealing with particular aspects of the theory of probable errors. A chronological list of such papers may well provide both a concise means of indicating this development, and a useful guide for those who desire references to a special topic. It is not claimed that this list is exhaustive, but it is believed that the more important works are mentioned.

The more important texts in statistical theory, and in particular the texts in the theory of probability, contain some work on probable errors, although few of them make original contributions on the subject. Certain fundamental papers of Edgeworth, Pearson, Sheppard, and others, on the law of error are not included in this list because they are not concerned *primarily* with the probable errors of frequency constants. These papers, nevertheless, contain much essential material on the topic. In general the most likely source for the formula for the probable error of any given measure is the memoir in which that measure was first employed.

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CHAPTER III

MOMENTS

1. WRITERS BEFORE PEARSON

Analogy to Mechanics. The analogy between the statistical function obtained by taking the sum of the nth powers of the deviations in any given distribution, and the mechanical concept of a *moment*, had been noticed by other writers before Pearson, but none of them had perceived in that function the powerful tool which he has developed.[1]

Laplace. In 1778 Laplace gave two formulas from which the numerical value of any even moment of the error function might be found.[2]

Kramp. The physicist Kramp, in his work on refraction,[3] gave numerical values of the first four even moments and a general formula for any even moment of the normal curve, the latter formula probably taken from Laplace.

[1] Pearson's nth moment around the mean is defined as $N\mu_n = \Sigma x^n$; the nth moment coefficient, as $\mu_n = \frac{\Sigma x^n}{N}$. A moment taken around any reference point other than the mean is distinguished by being marked with a prime. Thus μ_n' is the nth moment coefficient around an arbitrary origin.

Some other writers have used the terms *total moment* and *unit moment* for Pearson's *moment* and *moment coefficient*. Kelley uses *moment* where Pearson would use *moment coefficient*.

[2] See "Mémoire sur les Probabilités," *Histoire de l'Académie Royale des Sciences*, 1778 (published 1781). On page 292 is the formula

$$\int t^{2n}\,dt \cdot e^{-t^2} = \frac{1\,.\,3\,.\,5\,..\,(2n-1)}{2^n} \int dt\,.\,e^{-t^2},$$

and on page 193 the solution for finding the numerical value of

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt.$$

[3] *Analyse des Réfractions Astronomiques et Terrestres* (1799), p. 66.

Gauss. Gauss used the symbol $K^{(n)}$ as the most probable value of $\frac{\Sigma x^n}{N}$, the sign of x being always treated as positive.[4] Thus $K^{(n)}$ is comparable to the moment coefficient when n is even. Gauss also stated the expression for the probable error of a moment and gave numerical values of $K^{(n)}$ for the error function. He also made use of $\varepsilon_n = \sqrt[n]{K^{(n)}}$, a general expression similar to our $\sigma = \sqrt{\mu_2}$, and found the numerical values for the probable limits of ε_n for $n = 1$ to $n = 6$. He pointed out that ε_n is the most reliable when $n = 2$ (that is to say the standard deviation is more reliable than the mean deviation or than any other measure of this type). He also said that 100 observations would serve to determine ε_2 with the same degree of precision which could be obtained for ε_1, from 114 observations, for ε_3 from 109, for ε_4 from 133, for ε_5 from 178, and for ε_6 from 151.

Encke. The astronomer Encke developed the theory concerning the quantities $K^{(n)}$ and ε_n somewhat more fully.[5] He showed that the "middle error," which is our standard error, of $K^{(n)}$ is equal to $\sqrt{\frac{K^{(2n)} - K^{(n)}\,K^{(n)}}{m}}$.

[4] "Bestimmung der Genauigkeit der Beobachtungen," *Zeitschrift für Astronomie,* I (1816).

Gauss said nothing in this paper to indicate whether errors are measured from the arithmetic mean, but we know that in general he based his derivation of the curve of error on the assumption—quite definitely stated in *Theoria Motus*—that the arithmetic mean is the most probable measure of the true value of the unknown.

His reasoning is as follows: Let $\phi\,(x)$ be any law of distribution for the variable x, such that $\int_{-\infty}^{\infty} x\,dx = 1$, and let $\theta\,(x)$ be the area under the curve of distribution between the limits $-x$ and $+x$, deviations being presumably measured from the mean. Let $K^{(n)} = \int_{-\infty}^{\infty} \phi\,(x)\,x^n dx$. Then $K^{(n)}$ is the most probable value of

$$S^{(n)} = \alpha^n + \beta^n + \gamma^n + \delta^n + \text{etc.},$$

$\alpha, \beta, \gamma, \delta, \ldots$ being the errors of observation. The probability that the true value of $S^{(n)}$ lies between $K^{(n)} - \lambda$ and $K^{(n)} + \lambda$ is then

$$\Theta \frac{\lambda}{\sqrt{2\,m\,[K^{(2n)} - K^{(n)}\,K^{(n)}]}}.$$

The result is general for any law of distribution.

[5] "Über die Methode der kleinsten Quadrate," *Berliner Astronomisches Jahrbuch für 1834,* Berlin, 1832. Encke says that most of the material in the memoir is due to Gauss.

This follows from the proof of Gauss and is comparable to the formula

$$\sigma_{\mu_q} = \sqrt{\frac{\mu_{2q} - (\mu_q)^2}{N}}$$

Encke also gave a difference equation by which any $K^{(n)}$ could be computed from a K of lower order;[6] on the assumption of normality he found the values of $K^{(n)}$ from $n = 1$ to $n = 9$; and presented the general formulas for computing any ε_n:

$$(\text{for } n \text{ even})\ \varepsilon_n = \frac{1}{h}\sqrt[n]{\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdots (n-1)}{2 \cdot 2 \cdot 2 \cdot \cdots \cdots \quad 2}}$$

$$(\text{for } n \text{ odd})\ \varepsilon_n = \frac{1}{h}\sqrt[n]{\frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdots \frac{1}{2}(n-1)}{\sqrt{\pi}}}$$

Czuber. The treatment given by Czuber in his work on the theory of errors of observation[7] in 1891 follows Gauss very closely, even reproducing his values for the probable errors of ε_n.

Quetelet and De Forest. Quetelet used the idea of the lever to illustrate the arithmetic mean, which he called the center of gravity,[8] and De Forest[9] used the term *moments* in several papers.

2. PEARSON AND HIS ASSOCIATES

Pearson's First Use of the Term. Pearson's first published use of the term "moment" as applied to statistics seems to be in a letter to *Nature*,

[6] $K^{(n)} = \frac{\frac{1}{2}(n-1)}{h^2} K^{(n-2)}$. As usual, $h = \frac{1}{\sigma\sqrt{2}}$.

[7] *Theorie der Beobachtungsfehler.*

[8] "Sous le point de vue de la mécanique, les moments statiques . . . . se font donc équilibre autour du point extrême de la moyenne M, qui répond au centre de gravite," *Lettres sur la Théorie des Probabilités appliquée aux Sciences Morales et Politiques*, Brussels, 1846. Letter xvii, p. 392.

[9] Erastus de Forest, 1834-1888, graduate of Yale and of the Sheffield Scientific School, taught for some time in Melbourne, Australia, and spent the latter part of his life in Connecticut. He wrote several papers on the law of error, which were published either in *The Analyst* (Des Moines) or in the *Transactions of the Connecticut Academy of Arts and Sciences.* For further details see the list of his writings in the *Catalogue of Scientific Papers of the Royal Society*, and the article by Wolfenden "On the Derivation of Formulae for Graduation of Linear Compounding," *Transactions of the Actuarial Institute of America*, XXVI (1925). For his use of moments, see his article "On a Theorem in Probability," *The Analyst*, VII (1880), 169–176.

dated Oct. 26, 1893, where he said, "Accordingly I proceed *not* by the method suggested by Prof. Edgeworth's 'Law of Error and Elimination by Chance' . . . . but by a method of higher moments." In this letter he established the notation μ_n and suggested that the ratio of μ_3 to μ_2 might be used as a measure of the asymmetry of the curve. In the first of his "Mathematical Contributions to the Theory of Evolution" read on November 9th of that year, he used the term as though it were already well-known to his readers,[10] perhaps because he thought the analogy sufficiently clear to be used without explanation. In this paper he established the distinction between the notation μ_n for the nth moment around the mean, and μ'_n the nth moment around any arbitrarily chosen point. He gave here, probably for the first time, the formulas for computing the moments about the mean from the moments about an arbitrary origin, and described a graphical device for computing moments, which he said "has long been of use in graphical statics for finding the first, second, and third moments of plane areas."

The second paper in his series of "Contributions to the Mathematical Theory of Evolution"[11] defined the betas[12] and stated the relations between them which would produce each of the five types of frequency curves which he defined in that paper.

Sheppard's Corrections. The effect which grouping data in class intervals produces upon the moments was discussed by Sheppard in 1897.[13] He gave then without proof, the formulas known as "Sheppard's Corrections" for the first five moments, and said that the formulas apply only when (1) the size of the class interval is constant, (2) the distribution is practically

[10] "On the Dissection of Asymmetrical Frequency Curves," *Philosophical Transactions*, A, CLXXXV (1895), Part I, 71–110. (Read in 1894.)

[11] "Skew Variation in Homogeneous Material," *Philosophical Transactions*, A, CLXXXVI, (1895), Part I, 343–414.

[12] The betas are functions of the moments of a frequency distribution.

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \qquad \beta_3 = \frac{\mu_3\,\mu_5}{\mu_2^4} \qquad \beta_5 = \frac{\mu_3\,\mu_7}{\mu_2^5}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \qquad \beta_4 = \frac{\mu_6}{\mu_2^3} \qquad \beta_6 = \frac{\mu_8}{\mu_2^4}$$

[13] "On the Calculation of the Average Square, Cube, &c of a large number of Magnitudes," *Journal of the Royal Statistical Society*, LX (1897), 698–703.

continuous, and (3) the frequencies taper off gradually in both directions. He discussed the same problem[14] more exhaustively the next year.

A proof for these corrections was given by Pearson in 1904.[15] Here he used the symbol ν_n for the uncorrected moments and μ_n for the values obtained after the application of Sheppard's corrections. He pointed out that since mathematically the function and all its derivatives must vanish at the ends of the range, the formulas can properly be used only when contact is of an indefinitely high order, or, in other words, when the curve approaches the X axis as an asymptote.

Methods of Computation. Several writers, among them Elderton,[16] Sheppard,[17] Hardy,[18] and Czuber[19] have written on methods of computing the rough moments from actual statistics.

Two papers by Tchuproff "On the Mathematical Expectation of the Moments of frequency Distributions," translated from the Russian by Isserlis, have recently (1920, 1921) appeared in *Biometrika.*

Probable Error of a Moment. The effect of sampling upon the moments of a frequency distribution has been extensively studied by Pearson and some of his associates. The fundamental procedure was established by Pearson and Filon in 1898.[20] They set down here the general method by which the sampling error of any moment, whether it is a moment of the distribution of one trait, or of more than one, might be studied.[21] Among the specific formulas derived in this paper are the formulas for the standard deviation of a standard deviation (about which they say "The result is of

[14] "On the Calculation of the Most Probable Values of Frequency Constants for Data arranged according to Equidistant Divisions of a Scale," *Proceedings London Mathematical Society*, XXIX (1898), 353–380.

[15] "On an Elementary Proof of Sheppard's Formulae for correcting Raw Moments and on other Allied Points," *Biometrika*, III, 308–312.

[16] Elderton, "Notes on Statistical Processes," *Biometrika*, IV (1906), 374.

[17] Sheppard, "The Calculation of Moments of a Frequency Distribution," *Biometrika*, V (1907), 450–459.

[18] Hardy, *British Offices Life Tables.* Elderton (*Frequency Curves and Correlation* 1906, p. 19) gives Hardy credit for the summation method of computing moments.

[19] Czuber, *Die Statistischen Forschungsmethoden*, p. 62 (Vienna, 1921).

[20] "Mathematical Contributions to the Theory of Evolution.—IV. On the Probable Errors of Frequency Constants and on the Influence of Random Selection on Variation and Correlation," *Philosophical Transactions*, A, CXCI (1898), Part I, 229–311.

[21] Such expressions as $\Sigma\, xy$, or more generally $\dfrac{\Sigma\, x^q x^{q'}}{N} = p_{qq'}$, are called "product moments."

course old"), the standard deviation of a coefficient of correlation,[22] the correlation between sampling errors in the standard deviations of two correlated traits, and the correlation between sampling errors in a coefficient of correlation and in a standard deviation. The last three results were presented in this paper for the first time. Some of their statements are of such general application that we will quote them at some length:

"Attention should be drawn to the fact that we have replaced errors by differentials. This is only legitimate so long as product terms in the errors are negligible as compared with linear terms. This is the assumption almost universally made by writers on the theory of errors. [Footnote: Gauss, admittedly, 'Theoria Combinationis Observationum' . . . . . p. 53, Problema; Laplace and Poisson, actually but obscurely; see 'Théorie analytique des Probabilités, Liv. II, chap. IV., and 'Recherches sur la Probabilité des Jugements,' chap. IV.; more clearly in Todhunter's account, 'History of Theory of Probability,' Art. 1,002 *et seq.* Further Crofton, Article 'Probability,' §48 for a like assumption.] It will not lead us astray so long as we take care in any practical application to verify the smallness of Δr_{12}, $\Delta\sigma_1$, $\Delta\sigma_2$, as compared with r_{12}, σ_1 and σ_2 respectively," (p. 246.)

"The distribution of the errors of frequency constants, if treated exactly, will generally be skew, for the cubic and higher terms in the Δ's do not vanish. If however, the cubic terms are small as compared with the square terms, the frequency distribution of errors will approximate closely to a normal correlation surface. Probably the series in most cases converges with considerable rapidity. The fact, however, that we are dealing with the first terms of a series should be borne in mind. It does not seem to have been sufficiently emphasized when the probable error of the standard deviation is taken to be $67.449/\sqrt{2n}$ per cent of the standard deviation.

"Supposing the errors so small that we may neglect the cubic terms, we conclude that the errors made in calculating the constants of any frequency distribution are—

[22] Pearson had previously given, in his paper "Heredity, Regression, and Panmixia," the value $\dfrac{1 - r^2}{\sqrt{n(1 + r^2)}}$ as the standard error of a coefficient of correlation. In this paper of 1898, Pearson says that at the time he gave the earlier formula he "had not fully realised the importance of the principle of the correlation of errors made in determining the magnitude of frequency constants." He also presents here a table showing the difference between the "absolute" and "partial" probable errors of r. The "partial" probable error, that is the one which contains the factor $(1 + r^2)$ under the radical in the denominator, was used by Spearman as late as 1906.

"(i) Themselves distributed according to the normal law of errors.

"(ii) Correlated among themselves," (p. 234).

In 1903 Pearson published[23] formulas for the standard deviation of a moment taken about an arbitrary point, the standard deviation of a moment taken about the mean, and the correlation between two moments around arbitrary or true means.

Probable Errors of a Mean and of a Standard Deviation. The portion of this paper which touches most closely statistical practice as it occurs commonly in the field of education, is that which presents the formulas for the standard error of a mean and of a standard deviation.[24] As a special case of the standard error for any moment, he finds that the standard error of a mean is $\frac{\sigma}{\sqrt{m}}$, and that the standard error of a standard deviation is $\sqrt{\frac{\mu_4 - \mu_2^2}{4\mu_2 m}}$, m being the number of cases in the sample. Now if the kurtosis of the distribution is such that μ_4 is approximately equal to $3\mu_2^2$, as it is, for example, in a normal distribution, then the standard error of the standard deviation reduces[25] to the simpler form $\frac{\sigma}{\sqrt{2m}}$. It must not be supposed that the formulas for these probable errors were unknown before 1898. The formulas for measuring the reliability of a mean and of a standard deviation of a normal distribution had, in fact, been in use for many years, but so far as the writer knows, there had been before the time of Pearson and Sheppard, no comprehensive study of the reliability of the constants of skew distributions.[26] We have seen that Gauss (1816) and Encke (1832)

[23] "On the Probable Errors of Frequency Constants," Editorial in *Biometrika*, II (1903), 273 *et seq.*

[24] The mean $= \mu_1'$, and $\sigma = \sqrt{\mu_2}$.

[25] The actual practice in education is frequently to overlook the fact that when the distribution departs widely from the normal in its form, the shorter formula may not be appropriate. It cannot be considered appropriate unless $\beta_2 - 3$ is approximately zero.

[26] It is possible that Gram had done some work of this nature. Arne Fisher says that "J. P. Gram was the first mathematician to show that the normal symmetrical Gaussian error curve was but a special case of a more general system of skew frequency curves which could be represented by a series. In his very original doctor's thesis in Danish on the development of series by means of the method of least squares (Copenhagen, 1879) he extended some theories originally expounded by the Russian mathematician, Tchebycheff, to the representation of frequency functions by means of a series. . . . . To Gram, therefore, belongs the honor of having been the first mathematician to give a systematic theory for the development of skew frequency cruves." *Mathematical Theory of Probabilities*, pp. 182–183.

could compute the probable error of any even moment. At least as early as 1823, Gauss measured the variability of his middle error (standard deviation) and of his *error probabilis*, first deriving a form[27] comparable to Pearson's $\sqrt{\dfrac{\mu_4 - \mu_2^2}{4\mu_2 m}}$.

De Morgan made use of the standard error of the mean about 1838, in a paper[28] which presented a summary of the mathematical theory of probability as developed at that time. Airy stated the same rule as follows:

$$\text{"e·m·s·}^{29}\text{ of a measure} = \sqrt{\frac{\text{sum of squares of apparent errors}}{n-1}}$$

$$\text{e·m·s of the mean} = \sqrt{\frac{\text{sum of squares of apparent errors}}{n\,(n-1)}}$$

$$\text{p.e. of the mean} = 0.6745 \times \sqrt{\frac{\text{sum of squares of apparent errors}}{n\,(n-1)}}$$

The $(n - 1)$ enters the denominator on the assumption that the mean of the sample does not coincide with the true value of the quantity measured. Neither of these treatises contains any suggestion of the idea of moments as utilised by Encke and Gauss, or of a formula for the standard error of a standard deviation; nor has the writer found these in any other paper written in English before 1893.

[27] The derivation given by Gauss is as follows: Let $y = \dfrac{xx + x'x' + x''x'' + \text{etc.}}{\sigma}$ where σ is the number of cases. [i.e. $\sigma = N$.] Let the mean value of this be mm [m = our σ]. The actual value of this y will deviate more or less from its mean, but the chance that a value of y should deviate from its mean by a given amount grows less as σ increases. We desire to know the *errorem medium metuendum.* This error will be the square root of the mean of all values of $\left(\dfrac{xx + x'x' + x''x'' + \text{etc.}}{\sigma} - mm\right)^2$. If $\Sigma \dfrac{x^4}{\sigma^2} = \dfrac{n^4}{\sigma^2}$ and $\Sigma \dfrac{x^2}{\sigma} = m^2$, then he says we may show the mean of this function to be $\dfrac{n^4 - m^4}{\sigma}$. Previously he had shown (p. 12) that $n^4 = \frac{3}{4}\, h^4 = 3\, m^4$ if $x = \dfrac{e}{a\sqrt{\pi}}^{-\frac{x}{h}\cdot\frac{x}{h}}$. Therefore the mean error will be $mm\sqrt{\dfrac{2}{\sigma}}$.

[28] Article on "Probability" in the *Encyclopaedia Metropolitana.*

[29] e.m.s. = error of mean square.

Generalization of Theory of Errors in Moments. In 1898 Pearson and Filon gave the general theory of the probable errors of the constants of skew frequency distributions,[30] and in 1902 Pearson gave general expressions for the probable errors of β_1 and β_2, for $R_{\beta_1\beta_2}$, and for the probable error of the criterion K_2. These involve a knowledge of the higher betas, and these in turn require higher moments up to the 8th, which are subject to large sampling errors. In 1910 Rhind[31] presented new methods of obtaining these higher betas from given values of β_1 and β_2 stating two difference equations for computing β_n from β_{n-1} and β_{n-2}, one equation to be used when n is even and the other when n is odd. Ten tables for which β_1 and β_2 are the arguments are first presented in this paper. These give the value of $\frac{\sigma}{\sqrt{N}}$ for β_1, for β_2, for the skewness, for the modal divergence[32] of the curve, and for the criterion, the values of the higher betas up to the sixth, the correlation between errors in β_1 and β_2, the semi-major and semi-minor axes of the probability ellipse, and the angle between the major-axis and the axis of β_2. Three diagrams make it possible to determine by graphic methods the type of curve to which a distribution conforms, and the probability that this determination is satisfactory.

In 1913 Pearson returned to the subject of the probable errors of frequency constants,[33] and published probable errors and correlations of the higher product moments. His earlier paper[34] had dealt with moments of a single variable only. In this paper he uses the term "moment coefficient" for $\frac{\Sigma x^n}{N}$ as opposed to "moment" for Σx^n, and uses the symbol $p_{qq'}$ for $\frac{\Sigma x^q y^{q'}}{N}$. These terms and symbols, however, were used in some of his earlier papers.

[30] "On the Probable Errors of Frequency Constants and on the Influence of Random Selection on Variation and Correlation," *Philosophical Transactions*, A, CXCI (1898), 229–311.

[31] "Tables to facilitate the computation of the Probable Errors of the Chief Constants of Skew Frequency Distributions," *Biometrika*, VII, 127–147, 386–397. These tables have been incorporated in Pearson's *Tables for Statisticians and Biometricians*, 1914.

[32] The modal divergence is the difference between mean and mode.

[33] Editorial, "On the Probable Errors of Frequency Constants" II, *Biometrika*, IX (1913), 1–10. Pearson says that the material in this paper is taken from his lecture notes.

[34] *Biometrika*, II.

Some of the simpler outcomes of this investigation which are of immediate importance in the practice of educational statistics are the following:

(a) The correlation between means is the same as the correlation between the original measures if the sampling is random.

(b) For most practical purposes, the correlation between errors in σ_x and errors in σ_y will be equal to r^2_{xy}. This is true when both distributions are homoscedastic and have rectilinear regression.[35]

(c) The standard deviation of a coefficient of correlation is approximately

$$\sigma_{r_{xy}} = \frac{1 - r^2_{xy}}{\sqrt{N}} \left\{1 - \tfrac{1}{4} (\beta_2 - 3 + \beta_2' - 3) \frac{r^2_{xy}}{1 - r^2_{xy}}\right\}^{\frac{1}{2}}$$

where β_2 refers to one variable and β_2' to the other. If kurtosis is zero for both traits, then

$$\beta_2 - 3 = 0 = \beta_2' - 3,$$

and

$$\sigma_{r_{xy}} = \frac{1 - r^2_{xy}}{\sqrt{N}},$$

which is the value derived by Pearson and Filon for normal correlation surfaces. Pearson points out these implications:

"(i) Equal kurtosis is needful in the two variates if the regression is to be linear and the arrays homoscedastic in the case of each variable.

"(ii) The ordinary value subject to (i) is only correct provided the kurtosis is zero, and this is true whether the distribution be Gaussian or not.

"(iii) The ordinary formula may give very inaccurate results, if the kurtosis be considerable and the correlation high.

"(iv) It is probable that [the longer formula for r_{xy} stated above] . . . . gives fairly good results when the correlation is not linear.

"(v) There is a superior limit to the correlation which can be obtained in homoscedastic systems with linear regression, and a given kurtosis."

In the same paper, Pearson derives the probable error of a regression coefficient, of a regression value, and the constant term of a regression equation, and proves that for all symmetric systems there is no correlation between the mean and the standard deviation, and gives several related theorems.

[35] Pearson and Filon in 1898 found this to be true for cases of normal correlation.

Use of Moments in Curve Fitting. From the moments are derived the parameters in the Pearson system of curves.[36] The semi-invariants of Thiele,[37] $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots$ are similar in meaning to the moments, but the manner in which they are used in curve fitting is entirely different. Pearson's curves are all derived from a differential equation whose parameters are obtained from the moments of the distribution. The system of curves favored by the Scandinavian school of statisticians is derived from the expansion of an infinite series of the form

$$F(x) = c_0\,\phi(x) + c_1\,\phi'(x) + c_2\,\phi''(x) + c_3\,\phi'''(x) + c_4\,\phi''''(x) + \ldots$$

where the coefficients are functions of the moments. There are two types of these curves. Type A was introduced by the Danish mathematician J. P. Gram in 1879 in a work on series[38] in which he extended some theories of the Russian mathematician Tchebycheff. In the Type A curves the generating function $\phi(x)$ is the normal error-function. Fisher says that Gram was the first mathematician to give a systematic theory of the development of skew frequency curves. Somewhat later Thiele began to lecture at the University of Copenhagen on the theory of observations, and in 1889 published a book on that subject,[39] in which by means of his semi-invariants he arrived at the same series which Gram had used.[40]

Still later, Charlier[41] introduced a second form of curve to care for distributions that are markedly skew. In this Type B curve the generating

[36] The fundamental papers on this subject are three papers in Pearson's series of "Mathematical Contributions to the Theory of Evolution," as follows:

"II. Skew Variation in Homogeneous Material," *Philosophical Transactions*, A, CLXXXVI (1895), Part I.

"X. Supplement to a Memoir on Skew Variation," *Philosophical Transactions*, A, CXCVII (1901).

"XIX. Second Supplement to a Memoir on Skew Variation," *Philosophical Transaction*, A, CCXVI (1916).

[37] Thorwald Nicolai Thiele, 1838–1910, was until 1906 professor of astronomy and director of the observatory at the University of Copenhagen. He was instrumental in establishing a mathematical society (Mathematisk forening) in 1872 and an actuarial society (Aktuar foreningen) in 1901.

[38] *Om Raekkeudviklinger.*

[39] *Almindelig Iagttagelslaere.*

[40] These statements are based on *The Mathematical Theory of Probability*, by Arne Fisher, and on a conversation with Mr. Fisher.

[41] Karl Wilhelm Ludwig Charlier, born in 1862, was observer in the astronomical observatory at Stockholm, 1889–90, at Uppsala, 1890–97, and at Lund since 1897. In 1924 he was invited to the United States where he gave lectures in the University of California. For further biographical details, see the *Nordisk Familjbok*, IV.

function is not the normal error-function. In 1905 and perhaps earlier he described these two types of curve, adding the comment "Beyond these two forms no other frequency curves can occur, except those obtained through a superposition (addition) of several such curves of the types A and B."[42] It is obvious that the adequacy of this treatment depends upon the rapidity with which the series converge, and upon the precision of the higher moments which are known to be susceptible to very large sampling errors.

[42] "Researches into the Theory of Probability," *Acta Universitatis Lundensis*, I (1905), 5.

CHAPTER IV

Percentiles

1. early writers

Use by Various Persons. The discovery of the importance of the median as a measure of central tendency was made about fifty years ago by two different men who appear to have arrived at that measure independently and by widely diverse methods of approach. These two procedures will be treated in some detail in the following paragraphs. First, however, mention must be made of a few earlier writers who had conceived of this measure but who did not place great emphasis upon it.

Gauss. In 1816[1] Gauss suggested as a simpler method of computing the probable error the following: "Let all the errors be arranged in order of size without regard to sign, and take the middle one if the number is odd, or the mean of the two middlemost if the number is even." This value, which he called merely M, is thus the median of the absolute value of the errors, and similar to, though not necessarily identical with, Galton's quartile deviation.

Encke. A somewhat more explicit statement[2] of the method was made by Encke, who—as usual—credited it to Gauss. His statement (translated and somewhat abridged) is substantially as follows: "It is possible to use in determining r [the probable error] a rule which has no direct connection with the size of the separate errors, but which merely assumes that half of the errors will exceed and half fall short of r. Therefore the errors are to be arranged in order according to their absolute values, and counted off, beginning with the smallest. Let m be the number of the observations. When m is odd, the error corresponding to the index $\frac{1}{2}(m + 1)$; or when m is even the arithmetic mean between the errors corresponding to the indices $\frac{1}{2}m$ and $(\frac{1}{2}m + 1)$, will be an approximate value for the probable error." The probable error of this approximate value[3] he gives as $\frac{0.786716}{\sqrt{m}}(r)$.

[1] "Bestimmung der Genauigkeit der Beobachtungen."

[2] "Über die Methode der kleinsten Quadrate," (1832), pp. 194, 195.

[3] This would be about 1.113 times as large as the probable error of the standard deviation.

Quetelet. In one of his letters on the theory of probability,[4] Quetelet proposed the very measure which Galton afterwards termed the quartile deviation. Quetelet wrote: "Suppose a certain magnitude to be measured 1000 times, and the 500 results differing the least from the mean to be selected. Then the half-difference between the largest and the smallest of these measures will be the modulus of precision or the probable error."

That Galton had read this passage and had quite possibly found in it the germ of the idea which he developed into his system of grades and deviates, is not unlikely. In 1874 Galton contributed the section on *Statistics* to the first edition of a handbook called *Notes and Queries on Anthropology, for the Use of Travellers and Residents in Uncivilised Lands*, brought out by the British Association. In this section, Galton advised travelers interested in anthropology to report the height of men at the middle and quarter points, and also out of 1000 men to report the height of the 20th, 90th, 910th, and 980th, and to mention the number of people measured. He suggested that an African chieftain might be willing to line up his men in the order of their height, and yet be averse to having them all measured in a way that would make it possible to determine the mean height. In this same section on statistics, Galton remarks that a satisfactory treatment of the use of the law of error in such matters was not available, but that "Quetelet's Letters on the Theory of Probabilities is perhaps the most suitable to the non-mathematical reader."

2. FECHNER

Fechner's Use of the Median. In contrast to the practical interest in anthropology which impelled Galton to use the median and related measures, the incentive which led Fechner[5] to discover the median seems

[4] *Letters addressed to H. R. H. the Grand Duke of Saxe Coburg and Gotha, on the Theory of Probabilities as applied to the Moral and Political Sciences*, translated by Downes (London, 1849), Letter XX, p. 154. The date of the original publication in French was 1846.

[5] Gustav Theodore Fechner (1801–1887) wrote on a great variety of subjects, scientific, literary, metaphysical. He was first a student of medicine and then wrote a work on electricity (1831). These two interests brought him into contact with the physiologist Ernst Heinrich Weber, whose name is associated with his in connection with the "Weber-Fechner Law;" and with the physicist Wilhelm Weber, who seems to have first called Fechner's attention to the need of experimental studies to establish his law. Under the pseudonym of Dr. Mises he wrote literary criticisms, poetry, and satirical essays on the comparative anatomy of the angels, and on a proof that the moon is made of iodine. He also wrote a number of metaphysical works very different in spirit from his contribution to statistics and psycho-physics. He is probably best known for his researches in psycho-physics, and has been called "the father of the new-psychology." He also made important studies in theoretical and practical statistics, and Wundt calls him "der Vater der Kollectivmasslehre."

to have been a theoretical interest in generalizing the measures of central tendency. In his memoir "Ueber den Ausgangswerth der kleinsten Abweichungssumme, dessen Bestimmung, Verwendung, und Verallgemeinerung,"[6] he gave a full treatment of the properties and computation of the median, which he called the *Centralwerth*, or *C*. His argument is as follows:

The arithmetic mean (*A*) has the property that the sum of the squares of the deviations is less when these are measured from it than when they are taken from any other reference point. There may well be another mean so defined that the sum of the first powers of the deviations taken without regard to sign is a minimum when they are measured from it. Let this mean be called *der Centralwerth*. In general, there is a series of points of reference, or *Potenz-mittelwerthen*, around which the sum of the nth powers of the absolute values of the deviation is a minimum, one reference point corresponding to each value of n. Of these *Potenz-mittelwerthen* the mean *A* and the median *C* are special cases. Fechner suggested that as the Gaussian law[7] depends on the law of the arithmetic mean, there may be other laws of distribution corresponding to these other means.

He also called attention to certain properties of the median which are important in the field of mass phenomena. If *C* is known, it is possible to tell for any given value of the variate whether more than half of the remaining members of the group exceed it. This information is not provided by the mean alone. Also the numerical value and direction of the difference between *A* and *C* may be taken as a measure of the asymmetry of the distribution. In this same paper Fechner defined the mode D (*der dichteste Werth*) as that value around which the individual measures cluster most thickly. If *C* and *A* do not coincide, then *D* does not coincide with either of them. In this case *C* lies between *A* and *D*. A rational theory of asymmetry might be built from the relations between the positions of these three measures of central tendency. (Fechner says that he has not carried the laborious empirical investigations far enough to be able completely to understand the relations.)

Computation of the Median. Fechner's directions for computing the median are satisfactory and complete, even dealing with the exceptional cases. He lays down the fundamental rule that counting from one end of the distribution must produce the same result as counting from the other

[6] *Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, mathemathisch-physische Classe*, XI (1878).

[7] Gauss based his derivation of the law of error on the assumption that the arithmetic mean is the most probable value of the quantity measured.

end,[8] and he illustrates methods of computation which will not conform to that principle. The rule he gives for computing the median when some of the class frequencies are larger than 1 is identical with the rule found on page 56 of Kelley's *Statistical Method* (1923) and with the rule given by McCall in "How to Compute the Median," *Teachers College Record,* XXI (1920), 124–138. Fechner wrote his formula $C = g_1 + \dfrac{\frac{m}{2} - v}{z_0} I$,

where g_1 = initial point of the interval in which C lies,
z_0 = frequency in this interval,
v = frequency below g_1,
and I = size of the interval.

Fechner also stated that when the median falls in an interval between two measures, any point in that interval will satisfy his definition[9] of the median, and therefore the choice is an arbitrary one. Thus when m is even, the use of the $\dfrac{m+1}{2}$ measure as the median is arbitrary.

Fechner also suggested that from this computation of C there followed a method of computing the probable error by counting. There is nothing in the paper to imply that Fechner knew of the work that Galton was doing at this time.

3. GALTON

Galton's First Statistical Work. Galton's study of *Hereditary Genius* (1869) led him to feel the need of a statisfactory method of reporting differences in achievement and intellectual ability. Partly because his conception of the possibility of measurement was so highly original and partly because his statements suggest the course of thinking which soon caused him to devise his method of percentiles, we will quote several brief passages from this book. "The theory of hereditary genius, though usually scouted, has been advocated be a few writers in past as well as in modern times. But I may claim to be the first to treat the subject in a statistical manner, to arrive at numerical results, and to introduce the 'law of deviation from an average' into discussions on heredity. . . . . The range of mental powers between . . . . the greatest and least of English intellects, is enormous. There is a continuity of natural ability reaching from one knows not what height, and descending to one can hardly say what depth.

[8] Even now, fifty years later, there are texts in statistics which contain rules that violate this fundamental principle.

[9] Any point in the interval will also satisfy Galton's definition.

SIR FRANCIS GALTON (1822–1911)

From the *Life of Francis Galton*, by the permission of Professor Karl Pearson

. . . . I propose in this chapter to range men according to their natural abilities, putting them into classes separated by equal degrees of merit, and to show the relative number of individuals included in the several classes. . . . . The method I shall employ for discovering all this, is an application of the very curious theoretical law of 'deviation from an average.' First I will explain the law and then I will show that the productions of natural intellectual gifts come justly within its scope. . . . . Here we arrive at the undeniable, but unexpected conclusion, that eminently gifted men are raised as much above mediocrity as idiots are depressed below it; a fact that is calculated to considerably enlarge our ideas of the enormous differences of intellectual gifts between man and man."

The idea of *grades* has its first published statement in this book. Galton placed the ablest men in each million in the highest grade, and the most stupid in the lowest grade, and then divided the remaining 999,998 into 14 classes, the average ability of each being separated from that of its neighbors by *equal grades*, thus forming a table concerning which he said, "The table may be applied to special just as truly as to general ability. It would be true for every examination that brought out natural gifts, whether held in painting, in music, or in statesmanship. The proportion between the different classes would be identical in all these cases, although the classes would be made up of different individuals, according as the examination differed in its purport." It is clear that Galton was here moving rapidly in the direction of his percentile scale, although even the concept of the median is not explicit as yet.

"Statistics by Intercomparison." The section on *Statistics* in the Handbook of 1874 has already been mentioned. On February 27th of that same year Galton delivered a lecture at the Royal Institution on his "Proposed Statistical Scale" and followed the lecture by a letter to *Nature*[10] on March 5th explaining his idea of *ranks*. The next year his "Statistics by Intercomparison, with Remarks on the Law of Frequency of Errors," appeared.[11] In this paper he says that his object ". . . . is to describe a method for obtaining simple statistical results which has the merit of being applicable to a multitude of objects lying outside the present limits of statistical inquiry, and which, I believe may prove of service in various branches of anthropological research." While the idea of the median is implied in this passage it has not yet been designated by that name. "The object then found to occupy the middle position of the series must possess

[10] See *Journal of the Anthropological Institute*, IV (1874), 136–7.

[11] *Philosophical Magazine*, 4th series, XLIX (1875), 33–46.

the quality in such a degree that the number of objects in the series that have more of it is equal to that of those that have less of it." (p. 34.) Galton also suggests that "The most convenient measure of divergency is to take the object that has the mean value, on the one hand, and those objects, on the other, whose divergence in either direction is such that one half of the objects in the series on the same side of the mean diverge more than it does, and the other half less. The difference between the mean and either of these objects is the measure in question, technically and rather absurdly called the 'probable error.'" Here too is the well-known *ogive curve* which Galton introduced into statistics, and which is now found in many educational studies. In this paper Galton stressed the importance of the measures at the quarter points of the ogive, but gave them no name.

The material of this memoir Galton rewrote in the most elementary of language and published in *Mind* in 1880, giving a very clear and very simple explanation of the terms first quartile, last quartile, octile, suboctile, and middlemost or medium.

In none of Galton's work does there seem to be any reference to Fechner. It appears that the two men were working quite independently, and at least at this period, each in ignorance of the work of the other.

Pearson[12] says of this memoir, "As far as I am aware, however, Galton was the first to endeavour to unriddle something of nature's method of working from the frequency distribution of a given variate. We may see now-a-days that his solution of 1874 is not valid, but we have to confess that we have not got much farther than he did."

4. PERCENTILE SCALES

Galton's "Scales of Merit." In assigning marks of excellence of any sort, whether prizes or test scores, the first requisite is the ability to arrange all the members of the group in an order of merit. One of Galton's earliest attempts[13] to do this scientifically resulted in a "scale of merit" for the men who obtain mathematical honors at Cambridge. The marks which he assigned ranged from below 500 to 8,000 by intervals of 500, the senior wrangler obtaining nearly twice as many marks as the second wrangler, and more than thirty-two times as many as the last man. It was similar

[12] *Life, Letters and Labours of Francis Galton*, II, 340. An account of Galton's statistical investigations much more complete than this present study is given in this work, and in Volume III, now in press.

[13] *Hereditary Genius*, p. 19.

problems which led him to develop his percentiles, his ogive curves,[14] and his "schemes of distribution,"[15] for, to quote Galton once more, "A knowledge of the distribution of any quantity enables us to ascertain the Rank that each man holds among his fellows, in respect to that quality. This is a valuable piece of knowledge in this struggling and competitive world, where success is to the foremost and failure to the hindmost, irrespective of absolute efficiency. A blurred vision would be above all price to an individual in a nation of blind men, though it would hardly enable him to earn his bread elsewhere. When the distribution of any faculty has been ascertained, we can tell from the measurement, say of our child, how he ranks among other children in respect to that faculty, whether it be a physical gift, or one of health, or of intellect, or of morals."[16]

The Values of First and Second Prizes. This problem of the measurement of the relative abilities of individuals, which he had sensed in 1869 and had approached by various methods in the intervening years, Galton attacked directly in 1902, publishing in the first volume of *Biometrika* an article on "The Most Suitable Proportion between the Values of First and Second Prizes." A certain sum is available for two prizes to be awarded in a contest. How should the sum be most suitably divided between the first and the second competitor? What ratio should a first prize bear to a second one? What should be the division if there are more than two competitors? The interest of the paper depends less upon the author's conclusion that when only two prizes are given, the first ought to be about three times the second, than on its bringing to light a new application of the law of frequency of error. It gave a quantitative evaluation of the great width of the intervals which may exist between two adjacent individuals near the ends of the range as compared with those near the mode. In a "Note on Francis Galton's Problem" in the same issue of *Biometrika*, Pearson generalized the theory for any form of distribution, assuming merely that the competitors are a random sampling, not necessarily conforming to the normal law, and pointed out the bearing of the results on human relations, the immense increase in individuality as we pass from the average to the exceptional man. In concluding, he remarks that this difference problem "marks a new, and very probably a most important, departure in statistical theory."

[14] See his "Statistics by Intercomparison", *Philosophical Magazine*, 4th series, XLIX (1875), *et seq.*

[15] See *Natural Inheritance*, p. 37, *et. seq.*

[16] *Natural Inheritance*, p. 37.

Grades and Deviates. In Volume V of *Biometrika* (1906–07), Galton presented a paper on "Grades and Deviates," with a table computed by Sheppard giving values of the deviates of the normal curve for each permille of frequency. If the entire area under the normal curve can be divided by ordinates into 1000 equal parts, each of these portions is a *permille* of frequency, or a *millesimal grade*, and the distance of each ordinate from the mean ordinate, that is $\frac{x}{\sigma}$, is a deviate. The deviate is an abscissa, the grade an area. When percentile scores are to be translated into deviation scores, or when marks reported in categories are to be transformed into a scale having equal units, this table is of great value. In the latter connection, the Kelley-Wood table described on page 63 has some features that are even more helpful although it is not carried out to as many decimal places as Sheppard's table. The general theory of grades and deviates, especially as related to correlation,[17] has been greatly amplified by Pearson.

Applications in Modern Educational Practice. Galton's general method of grades and deviates is widely used today in assigning class marks, and in constructing educational tests and scales.[18] It is fundamentally the method employed by McCall for the construction of a T-Scale. Rugg in his text on *Statistical Methods Applied to Education,* gave two tables of "percentile scores to be assigned to test problems or questions which correspond to various percentages of pupils who fail to solve problems or questions correctly." One of these tables assumes the normal probability curve to touch the base line at the points $\pm 2.5\sigma$ and the other assumes it to touch at $\pm 3\sigma$. Otherwise the tables agree with those prepared by Sheppard. McCall has used a different range in his T-Scales,[19] assuming that the curve comes down to the base line at the points $\pm 5\sigma$, so that the entire range is 10σ, which he sets equal to $100T$. Thus $10T = \sigma$. Like Rugg, he transfers the origin from the mean to the point at which he assumes the range to begin. Also like Rugg, he uses 50 as the T-Score for the mean.

[17] See, for example, his memoir on "On Further Methods for Measuring Correlation," *Drapers' Company Research Memoirs, Biometric Series,* IV, 1907, and numerous other studies listed in the bibliography on pages 130 to 141.

[18] No attempt is made to present a discussion of the history of tests and scales at the present time but the writer hopes to publish a treatise on this topic at a later date.

[19] See *How to Measure in Education* (1923), and also *How to Experiment in Education* (1923). "T" stands for Thorndike, and for Terman, and it might reasonably suggest *tenths*, as well.

For a normal distribution, the T-Score would be given by the formula

$$T = \frac{10x}{\sigma} + 50,$$

but instead of using this formula, which would imply the use of the method of moments, he computes his T-Scores by the percentile method just as Galton found deviates from grades. Both Galton's method of computing deviates from grades and McCall's T-Scale have been frequently misused by workers who fail to inquire into the normality of their distributions.

CHAPTER V

Correlation

1. PURPOSE OF THE CHAPTER

A complete history of correlation theory can scarcely be brought within the scope of this present study. It is indeed doubtful if an adequate history of that subject can be written by anyone now living except Professor Pearson. The published memoirs on correlation probably tell only part of the story, and need to be supplemented by his own recollections and unpublished notes and papers.

The purpose of the present treatment is to present a somewhat abbreviated and schematic account of the development of the theory of correlation and its applications, and to furnish references to sources from which more details may be obtained. It is hoped that this may set the topic in historical perspective for those who—not having access to a large library—find it impossible to refer to the original papers; and that it may assist those who wish to go to the sources in finding important articles with greater ease than would be otherwise possible.

Since correlation theory developed historically from the theory of probability, it has seemed necessary to make a brief reference to the mathematical equation for the probability surface of normal correlation. The non-mathematical reader who finds the next few paragraphs irksome, will find material more to his liking in Section 3.

2. WRITERS BEFORE GALTON

Idea of Correlation Due to Galton. It is generally understood that correlation was the unique discovery of Sir Francis Galton, made early in the last quarter of the nineteenth century, and greatly elaborated and refined by Pearson, Edgeworth, and Weldon, and by others who came later. Nevertheless there were others in that century who hovered on the verge of the discovery of correlation. Any one of them might have discovered it. None of them did. A French mathematician, an Italian astronomer, a German mathematician, and a French physicist each approached the problem from the point of view of mathematical analysis, but failed to see the practical significance of the formulas they derived. This may have been indeed because their attention was focused too closely upon the theory and not

enough directed toward empirical data. Several decades later an American physiologist saw clearly the need for a measure of the relationship between height and weight, but was unable to devise a satisfactory formula. Their work will be described briefly before the actual discovery by Galton is considered.

Correlation Based on Probability. To understand the early work on correlation, we must recognize the intimate relation between correlation theory and the theory of probability. The equation for ordinates of the probability surface of normal correlation for two variables is:

$$z = \frac{1}{2\,\sigma_1\,\sigma_2\,\pi\sqrt{1-r^2}}\,e^{-\frac{1}{2(1-r^2)}\left\{\frac{x_1^2}{\sigma_1^2}-\frac{2\,rx_1\,x_2}{\sigma_1\,\sigma_2}+\frac{x_2^2}{\sigma_2^2}\right\}} \tag{1}$$

The r in this formula is the familar coefficient of correlation.

The equation $\frac{x_1^2}{\sigma_1^2} - \frac{2\,rx_1\,x_2}{\sigma_1\,\sigma_2} + \frac{x_2^2}{\sigma_2^2} = K,$

where K is a parameter, is the equation for the family of contour lines of the normal correlation surface, that is for the lines in which the surface would be intersected by a set of horizontal planes. If $r = 0$, the contour lines are all ellipses whose axes coincide with the axes of x and y. If $\sigma_1 = \sigma_2$ and $r = 0$, these ellipses become circles. If r is neither zero nor 1, the contour lines are ellipses whose axes are oblique to the axes of x and y. As the numerical value of r increases these ellipses become more and more elongated, until when correlation is perfect—no matter whether positive or negative—the ellipses have collapsed into straight lines, all the points being now located on a single regression line of perfect correlation.

The term $\frac{-2rx_1x_2}{\sigma_1\,\sigma_2}$ in the exponent will be spoken of as the "product term." It is present in the exponent only when the two variates under consideration are correlated, and disappears whn $r = 0$. Normal correlation between two variates is a special case of the more general problem of skew correlation among n variates.[1]

Adrain, Laplace, Plana, Gauss, Bravais each discussed the probability of

[1] A treatment of "normal correlation" may be found in any of the following:

Yule, *Introduction to the Theory of Statistics*, Chapter XVI, "Normal Correlation."

Jones, *A First Course in Statistics*, Chapter XIX, "Frequency Surface for Two Correlated Variables."

Whittaker and Robinson, *The Calculus of Observations*, Chapter XII, "Correlation."

Coolidge, *Introduction to Mathematical Probability*, pp. 142–150.

the simultaneous occurrence of two errors, and the last four derived an equation for the probability surface containing the product term in the exponent. Czuber and De Forest also wrote on the subject, the latter following Bravais closely, but they added nothing significant to the theory. None of them saw that the strength of the relationship between two variates might be of major importance. None of them conceived of this as a matter which could have applications outside the fields of astronomy, physics, and geodesy or gambling.

Adrain. As has been mentioned previously,[2] Adrain published in 1808 two proofs of the law of error. The second of these is less elegant than the first, but is interesting because of the fact that it appears to have been the first study of the probability of the simultaneous occurrence of two independent errors in the position of a point. He did not consider the related question for dependent variables, but he recognized that "curves of equal probability" would be concentric circles for two equal "sources of error" and would be concentric ellipses when "the equally probable sources of error" are in any ratio other than one to one. These curves are exactly the contour lines of our surface of normal probability for uncorrelated variables, the ellipses being elongated in the direction of the variable with the greater dispersion. Adrain had no product term, and no interest in the relation between the variables except to specify that they should be independent.

Laplace and Plana. In 1810, in a paper[3] on definite integrals, Laplace developed the expression

$$\iint \frac{k}{4k''\pi} \cdot \frac{I}{\sqrt{E}} \cdot \frac{du \cdot du'}{a^2} \cdot c^{-\frac{k(Fu^2 + 2Guu' + Hu'^2)}{4k''a \cdot E}},$$

which was reprinted in his *Théorie Analytique* (1812), page 325. In the same year in which the latter work came out, Plana[4] read a paper in which

[2] See page 20.

[3] "Mémoire sur les intégrales définies et leur application aux probabilités," *Mémoires de l'Institut Impérial de France,* Année 1810 (1811), 279–347.

[4] "Mémoire sur divers problèmes de probabilité," *Mémoires de l'Académie Impériale de Turin, pour les* Années 1811–1812, XX, 355–498 (Turin, 1813). The paper was read Nov, 30, 1812. For a more complete analysis of the material in this paper, see Walker, "The Relation of Plana and Bravais to the Theory of Correlation," *Isis,* X (1928), 466–484.

While this is in press, Professor Pearson writes that he disagrees with the writer's interpretation of Plana's work, and that he is publishing in the forthcoming number of

he amplified some aspects of Laplace's work and gave the form

$$z = \frac{1}{4\,\pi\,a\,\sqrt{E}}\,e^{-\frac{1}{4\,a\,E}(CQ^2 - 2BQQ' + AQ'^2)}.$$

Comparing this with the formula (1), we see that the following correspondences are suggested: E with $1 - r^2$, A and C with the ratios between the two sigmas, a with one-half the product of the sigmas, and B with r in the usual formula for normal correlation. Plana also generalized the case to include the problem of three related variables. He did not attach any particular significance to the letter B, nor does he seem to have regarded the formula as of great importance. When writing on the subject of probability again in 1818 and in 1820,[5] he did not recur to this matter. Laplace reproduced the solution of 1810 in later editions of his *Théorie Analytique*, but without isolating the product term for particular attention, or recognizing that the strength of the relationship between the variables might be a matter of major importance. These papers by Laplace and Plana seem to have been the first to show the product term in the expression for the probability of the simultaneous occurrence of two variables, and the only ones before the time of Galton to use a single symbol comparable to our coefficient of correlation. No writer on correlation seems to have mentioned either Laplace or Plana in connection with that subject, and this memoir by Plana seems to be almost entirely unknown.[6]

Biometrika a critical note on the subject. The reader is referred to this criticism which the writer has not yet seen.

Giovanni Antonio Amedeo Plana (1781–1864) was a professor of astronomy and director of the observatory at Turin, and contributed many scientific papers to Italian, French and German societies. The *Catalogue of Scientific Papers of the Royal Society of London* contains a list of one hundred and thirty-three of his papers, only three of which, however, appear to have any bearing on probability. Some biographical material concerning Plana is to be found in a paper entitled "Éloge Historique de Jean Plana, l'un des huit associés étrangers de l'Académie" by M. Élie de Beaumont, read in public session of the Academy November 25, 1872. The nine letters now in the possession of Professor David Eugene Smith, written by Plana from Turin after the year 1828, contain no material related to probability, although they furnish some interesting personal details.

[5] "Allgemeine Formeln um nach der Methode der kleinsten Quadrate die Verbesserungen, von 6 Elementen zu bestimmen," *Zeitschrift für Astronomie und verwandte Wissenschaften*, VI (1818), 249–264.

"Soluzione generale di un problema di probabilità," *Memorie di Matematica e di Fisica della Società Italiana Delle Scienze, residente in Modena*, XVIII (1820), 31–45.

[6] Czuber gives the title in the very extensive bibliography at the close of "Die Entwicklung der Wahrscheinlichkeitstheorie, und ihrer Anwendung" (*Jahresbericht der Deutscher Mathematiker-Vereinigung*, VIII, Leipzig, 1899.) as does also Keynes in the bibliography in his *Treatise on Probability*, and Merriman includes it in his "List of Writings Relating to the Method of Least Squares," but indicates that he had not read it.

Gauss. In his *Theory of the Combination of Observations*[7] (1823) and the Supplementum (1826) Gauss developed the mathematics of the probability of the simultaneous occurrence of two or more errors. An analysis of Gauss's relation to the correlational calculus has been given by Pearson in his "Notes on the History of Correlation."[8] Referring to the formula which Gauss derived for the probability of the simultaneous occurrence of x_1, x_2, ... Pearson says: "This is a normal surface which contains the product terms. As we now interpret it we say that the x's are *correlated variates*. And in this sense Gauss in 1823 reached the normal surface of n correlated variates. But he does not seek to express all his relations in terms of the S.D.'s σ_{x_1}, σ_{x_2}, σ_{x_3}, ... and the correlations r_{12}, r_{13} . . . . of these variates. These x-variates are not for Gauss, nor for those who immediately followed him, the *directly* observed quantities. . . . . The point is this, that the Gaussian treatment leads (i) to a non-correlated surface for the directly observed variates, (ii) to a correlation surface for the indirectly observed variates. This occurrence of product terms arises from the geometrical relations between the two classes of variates, and not from an organic relation between the indirectly observed variates appearing on our direct measurement of them. It will be seen that Gauss's treatment is almost the reverse of our modern conceptions of correlation. . . . . In short there is no trace in Gauss's work of observed physical variables being—apart from equations of condition—associated organically which is the fundamental conception of correlation." (p. 13.)

Bravais. The name of Bravais[9] is familar to most students of statistics, although his writings are not easily accessible and have probably not been widely read in this country. The celebrated paper[10] which he read before the Institut de France in 1846 is often referred to as containing the first exposition of correlation theory, and it is known that he set forth the mathematics of the normal correlation surface three decades before the idea

[7] *Theoria Combinationis Observationum Erroribus Minimis Obnoxiae.*

[8] *Biometrika,* XIII (1920–21).

[9] Auguste Bravais (1811–1863) was a lieutenant in the French navy, professor of astronomy at Lyons, and professor of physics at Paris. He made scientific journeys and wrote many memoirs on astronomy, geodesy, optics, and physics. So far as the writer knows, none of his papers other than the one mentioned here treats of probability, although he is known to have had some correspondence with Quetelet on that subject.

[10] "Analyse mathématique sur les probabilités des erreurs de situation d'un point," *Mémoires de l'Institute de France,* IX (1846), 255–332.

of correlation had been conceived. That a similar result had been reached thirty-four years earlier by Plana has apparently not been pointed out by any previous writer.

The method by which Bravais reached the equation for the surface of normal probability is geometric, and is thus in direct contrast with the purely analytic treatment presented by Plana, who suggested no spatial conformations at all. Bravais recognized the existence of a relationship, a "correlation," between his principal variables,[11] but gave it merely passing notice.

The formula which Bravais derived for the ordinates of a surface of normal frequency for two variables was:

$$z = \frac{K}{\pi} e^{-(ax^2 + 2\,cxy + by^2)}.$$

When we compare these results with formula (1) , we see that

$$K = \frac{1}{2\,\sigma_1\,\sigma_2\,\sqrt{1 - r^2}},$$

$$a = \frac{1}{2\,\sigma_1^2\,(1 - r^2)},$$

$$b = \frac{1}{2\,\sigma_2^2\,(1 - r^2)},$$

$$c = \frac{-\,r}{2\,\sigma_1\,\sigma_2\,(1 - r^2)},$$

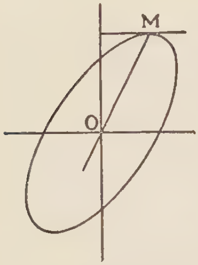
$$K^2 = \frac{1}{4\,\sigma_1^2\,\sigma_2^2\,(1 - r^2)^2}.$$

Obviously Bravais had no single term equivalent to our coefficient of correlation.

The significant portion of Bravais's paper, so far as correlation theory is concerned, is that he called attention to the fact that the equation $ax^2 + 2exy + by^2 = K$ represents an ellipse with center at the origin, and

[11] "La coexistence des mêmes variables m, n, p . . . . dans les équations simultanées en x et y, amène une corrélation telle, que les modules h_x, h_y, cessent de représenter la possibilité des valeurs simultanées de (x, y) sous le vrai point de vue de la question." (p. 263.)

principal axes in general different from the coordinate axes. If K be allowed to vary, there will arise an unlimited series of similar ellipses. He found the equation of the line OM, but never dreamed that it would one day achieve fame as a "line of regression." Bravais was unfortunately more concerned with the areas of the ellipses than with any relationship between the variables, and so remained unaware of the stupendous idea in whose vicinity his mind was hovering. That he might, with one leap of creative imagination, have pounced squarely upon this conception of a measure of the relationship between x and y, is apparent from his discussion on page 179. It is equally apparent that he did not leap. On this page he suggests than when the coordinates are transformed so that the axes of the ellipses coincide with the coordinate axes, the equation reduces to the form



$$z = \frac{1}{\pi}\sqrt{h_x h_y}\, e^{-(h_x x^2 + h_y y^2)},$$

h being equal to $\frac{1}{2\sigma^2}$. Upon this fact he makes the pertinent comment that in this case the probability is exactly the same as if the variables x and y were entirely independent one of the other.

He also gives the analogous case for three variables and observes that the locus of equally probable positions of the point are in this case a series of concentric ellipsoids. In his résumé, Bravais shows beyond the possibility of misunderstanding that he had no thought of applying his theory in any field other than that specified in the title—errors in the geometric position of a point.

Plana's paper seems to have been unknown to Bravais and to most other writers on probability; and Bravais's paper seems to have received little attention until after Galton published *Natural Inheritance* in 1889, although the American mathematician De Forest refers to it.

Bowditch. The study made by Bowditch on the *Growth of Children* in 1877 was an extensive statistical study on about 24,500 children. Record was made of certain bodily measurements, age, occupation of parents, nationality, birthplace, and remarks by the teacher. The study was sponsored by the Massachusetts Board of Health. One of its objects was

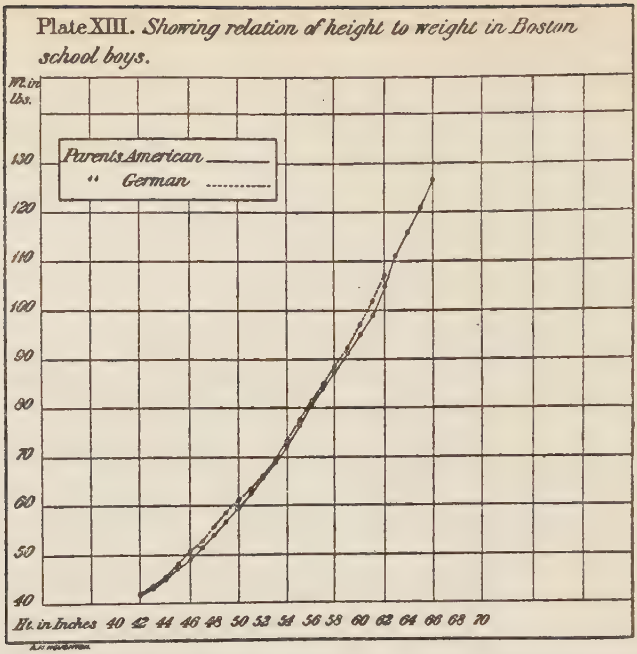
Table Showing Heights of Boston School Boys. Whole Number of Observations, Irrespective of Nationality

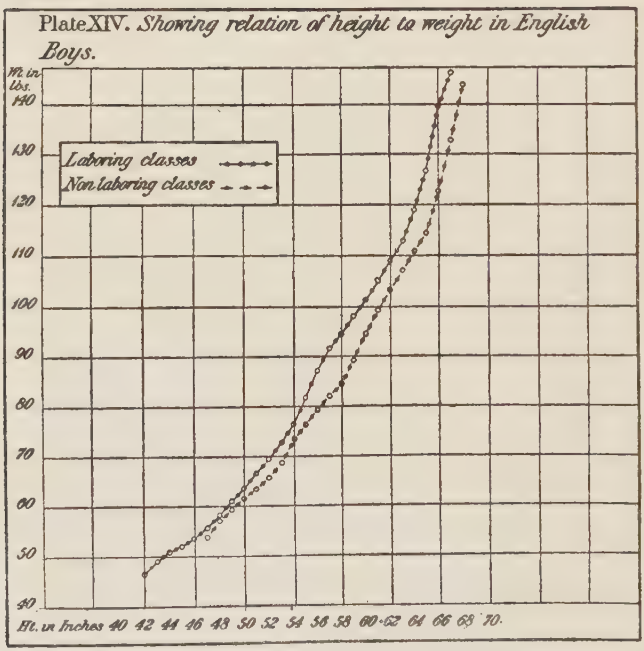
Inches	5 yrs. No.	5 yrs. Per cent	6 yrs. No.	6 yrs. Per cent	7 yrs. No.	7 yrs. Per cent	8 yrs. No.	8 yrs. Per cent	9 yrs. No.	9 yrs. Per cent	10 yrs. No.	10 yrs. Per cent	11 yrs. No.	11 yrs. Per cent
74	...	...	...	...	...	...	...	...	...	...	...	...	...	...
73	...	...	...	...	...	...	...	...	...	...	...	...	...	...
72	...	...	...	...	...	...	...	...	...	...	...	...	...	...
71	...	...	...	...	...	...	...	...	...	...	...	...	...	...
70	...	...	...	...	...	...	...	...	...	...	...	...	...	...
69	...	...	...	...	...	...	...	...	...	...	...	...	...	...
68	...	...	...	...	...	...	...	...	...	...	...	...	...	...
67	...	...	...	...	...	...	...	...	...	...	...	...	...	...
66	...	...	...	...	...	...	...	...	...	...	...	...	...	...
65	...	...	...	...	...	...	...	...	...	...	...	...	...	...
64	...	...	...	...	...	...	...	...	...	...	...	...	...	...
63	...	...	...	...	...	...	...	...	...	...	...	...	2	.15
62	...	...	...	...	...	...	...	...	...	...	...	...	1	.08
61	...	...	...	...	...	...	...	...	...	...	...	...	1	.08
60	...	...	...	...	...	...	...	...	...	...	1	.07	5	.39
59	...	...	...	...	...	...	...	...	1	.07	2	.15	12	.92
58	...	...	...	...	...	...	...	...	2	.14	4	.29	23	1.78
57	...	...	...	...	...	...	...	...	1	.07	13	.95	41	3.17
56	...	...	...	...	...	...	2	.13	4	.27	38	2.78	96	7.42
55	...	...	...	...	...	...	1	.07	7	.49	61	4.47	156	12.06
54	...	...	...	...	...	...	9	.61	24	1.66	115	8.44	187	14.46
53	...	...	...	...	...	...	6	.40	61	4.24	162	11.89	204	15.77
52	...	...	...	...	2	.14	24	1.62	113	7.86	226	16.58	192	14.85
51	...	...	...	...	7	.43	55	3.71	186	12.94	219	16.07	163	12.61
50	...	...	1	.07	18	1.27	123	8.31	252	17.54	208	15.26	119	9.20
49	...	...	7	.55	60	4.23	197	13.30	289	20.11	176	12.91	50	3.87
48	...	...	22	1.74	105	7.40	274	18.50	219	15.24	79	5.80	25	1.97
47	4	.47	47	3.73	202	14.24	303	20.46	163	11.34	41	3.01	6	.46
46	8	.94	96	7.63	270	19.03	225	15.20	73	5.08	12	.88	8	.62
45	20	2.35	170	13.51	270	19.03	150	10.13	27	1.89	3	.22	2	.15
44	62	7.31	253	20.19	251	17.69	71	4.79	8	.56	3	.22	...	...
43	119	14.03	260	20.66	126	8.88	32	2.16	3	.21	...	...	...	...
42	149	17.57	219	17.40	71	5.00	2	.13	1	.07	...	...	...	...
41	190	22.40	100	7.94	24	1.69	3	.20	1	.07	...	...	...	...
40	149	17.57	60	4.76	10	.70	2	.13	1	.07	...	...	...	...
39	79	9.31	14	1.11	2	.14	...	...	1	.07	...	...	...	...
38	42	4.95	7	.55	...	...	1	.07	...	...	...	...	...	...
37	17	2.00	1	.07	...	...	1	.07	...	...	...	...	...	...
36	7	.82	1	.07	1	.07	...	...	...	...	...	...	...	...
35	1	.12	...	...	...	...	...	...	...	...	...	...	...	...
34	...	...	...	...	...	...	...	...	...	...	...	...	...	...
33	...	...	...	...	...	...	...	...	...	...	...	...	...	...
32	...	...	...	...	...	...	...	...	...	...	...	...	...	...
31	...	...	...	...	...	...	...	...	...	...	...	...	...	...
30	1	.12	...	...	...	...	...	...	...	...	...	...	...	...
Totals	848		1258		1419		1481		1437		1363		1293	

12 yrs. No.	12 yrs. Per cent	13 yrs. No	13 yrs. Per cent	14 yrs. No.	14 yrs. Per cent	15 yrs. No.	15 yrs. Per cent	16 yrs. No.	16 yrs. Per cent	17 yrs. No.	17 yrs. Per cent	18 yrs. No.	18 yrs. Per cent	Inches
...	...	...	...	...	...	1	.16	...	...	1	.52	...	...	74
...	...	...	...	...	...	...	...	1	.28	1	.52	...	...	73
...	...	...	...	...	...	...	...	1	.28	3	1.56	...	...	72
...	...	...	...	...	...	2	.31	3	.84	4	2.08	3	3.57	71
...	...	...	...	1	.11	3	.47	11	3.06	10	5.21	6	7.14	70
...	...	...	...	6	.66	5	.79	14	3.90	13	6.77	5	5.95	69
...	...	1	.09	6	.66	18	2.83	17	4.73	25	13.02	11	13.09	68
...	...	5	.43	17	1.87	29	4.56	47	13.09	20	10.42	14	16.66	67
1	.08	2	.17	12	1.32	37	5.81	53	14.76	24	12.50	14	16.66	66
1	.08	10	.86	31	3.41	62	9.75	50	13.93	33	17.19	13	15.47	65
1	.08	11	.95	46	5.06	57	8.96	44	12.26	17	8.85	9	10.74	64
4	.32	19	1.64	49	5.39	62	9.75	41	11.42	17	8.85	5	5.95	63
9	.72	37	3.19	68	7.48	67	10.51	26	7.24	12	6.25	2	2.38	62
10	.50	59	5.08	75	8.25	58	9.12	19	5.29	4	2.08	1	1.19	61
31	2.47	67	5.78	101	11.12	72	11.32	11	3.06	3	1.56	1	1.19	60
49	3.91	106	9.11	105	11.56	66	10.38	13	3.62	3	1.56	...	...	59
77	6.15	125	10.78	127	13.98	36	5.66	4	1.11	1	.52	...	...	58
106	8.46	134	11.55	95	10.46	22	3.46	2	.56	...	...	...	...	57
162	12.93	192	16.55	62	6.82	16	2.52	2	.56	...	...	...	...	56
198	15.80	141	12.15	47	5.17	11	1.73	...	...	...	...	...	...	55
189	15.08	102	8.79	34	3.74	9	1.42	...	...	1	.52	...	...	54
176	14.04	79	6.81	11	1.21	2	.31	...	...	...	...	...	...	53
111	8.86	32	2.76	7	.77	1	.16	...	...	...	...	...	...	52
73	5.82	18	1.55	4	.44	...	...	...	...	...	...	...	...	51
31	2.47	12	1.03	3	.33	...	...	...	...	...	...	...	...	50
13	1.03	4	.34	...	...	...	...	...	...	...	...	...	...	49
7	.55	2	.17	1	.11	...	...	...	...	...	...	...	...	48
3	.23	2	.17	...	...	...	...	...	...	...	...	...	...	47
1	.08	...	...	...	...	...	...	...	...	...	...	...	...	46
...	...	...	...	...	...	...	...	...	...	...	...	...	...	45
...	...	...	...	...	...	...	...	...	...	...	...	...	...	44
...	...	...	...	...	...	...	...	...	...	...	...	...	...	43
...	...	...	...	...	...	...	...	...	...	...	...	...	...	42
...	...	...	...	...	...	...	...	...	...	...	...	...	...	41
...	...	...	...	...	...	...	...	...	...	...	...	...	...	40
...	...	...	...	...	...	...	...	...	...	...	...	...	...	39
...	...	...	...	...	...	...	...	...	...	...	...	...	...	38
...	...	...	...	...	...	...	...	...	...	...	...	...	...	37
...	...	...	...	...	...	...	...	...	...	...	...	...	...	36
...	...	...	...	...	...	...	...	...	...	...	...	...	...	35
...	...	...	...	...	...	...	...	...	...	...	...	...	...	34
...	...	...	...	...	...	...	...	...	...	...	...	...	...	33
...	...	...	...	...	...	...	...	...	...	...	...	...	...	32
...	...	...	...	...	...	...	...	...	...	...	...	...	...	31
...	...	...	...	...	...	...	...	...	...	...	...	...	...	30
1253		1160		908		636		359		192		84		

Age at last birthday.

BOWDITCH'S HEIGHT-AGE TABLE, 1877





Early Regression Curves of Weight on Height (1877)

From Bowditch's "Growth of Children" as reprinted by the American Statistical Association in *Papers on Anthropometry*

to improve school practice with respect to growing children. The section headed "Relation of Height to Weight" is particularly interesting. Desiring to measure the relation of height to weight as age varies, Bowditch devised the plan of quoting the ratio of the annual increase in pounds weight to the annual increase in inches height for each age from 5 to 18, a ratio which he calls the "measure of stoutness." In this section he was not far from the idea of correlation and not far from the idea of regression, but he was impeded by the fact that what he really was seeking was partial correlation. He wanted to know the relation between height and weight in general, and also for each separate year of age, and was confused by the attempt to consider both problems together.

Bowditch had six large charts which we would now call correlation charts. In three of them the height in inches is plotted against the age in years, and in the remaining three the weight in pounds is plotted against the age in years. No lines are drawn on these charts, but there is a separate series of remarkable graphs in which a line is drawn through the points which represent the average height (or weight) for each year of age. These are nothing short of curvilinear regression lines for height on age and weight on age. In four plates the regression of weight on height is also shown. In no case, however, did Bowditch draw *both* lines for a single table, and in no case did he superimpose the line on the correlation chart. The discussion on page 93 indicates that Bowditch felt an urgent need of some method of measuring the strength of the relationship between his variables and that he was dissatisfied with the method he adopted, which he said was defective, "first because it does not take into account the possible influence of age upon the ratio of a given height to its corresponding weight; and secondly because it rests upon the assumption that the average weight for a given age is the same as the average weight of all individuals, without regard to age, whose height is equal to the average height for that age."

Galton and Bowditch were in correspondence at this time, having been introduced through a letter of Charles Roberts, who suggested to Bowditch that Galton might be able to furnish him some information concerning racial types. On June 26, 1876, Galton wrote Bowditch, "I trust you will exclude no cases, however exceptional and that you will not trim your results to the Procrustean bed of the 'Law of Error' by only giving . . . . for the final results the mean and the assumed probable error,—but that you will give a fair number of ordinates, sufficient to enable the curve (what-

soever it may be) to be constructed and intermediate values to be found by interpollation."[12]

3. DISCOVERY OF CORRELATION BY GALTON

Sources of Information. Galton's own most delightful account of his discovery of regression—from which the concept of correlation developed—is to be found in his *Memories of My Life* (1908), in the chapter on Heredity. A statement of the theory as he had organized it in 1889 may be read in *Natural Inheritance*, chapter VII, "Discussion of the Data of Stature." A still more complete history, enriched by numerous quotations from Galton's unpublished letters and papers, is available in Pearson's *Life, Letters and Labours of Francis Galton*, Vol. II. Each of these books may be easily obtained, and each offers extremely good reading, even to the layman who has little previous knowledge of correlation theory. Because these accounts are not difficult to secure, the following summary of the events in Galton's momentous discovery has been confined within limits somewhat more narrow than would otherwise be consistent with the epoch-making character of that powerful concept. The reader who refers to the longer accounts will find them full of human as well as technical interest.[13]

Early Interest in Heredity. The publication in 1859 of Darwin's *Origin of Species* had a profound influence upon Galton, who says: "Its effect was

[12] This letter is in the possession of the Genetics Record Office, Carnegie Institution, Cold Spring Harbor, Long Island.

[13] An important and somewhat more technical treatment is to be found in Pearson's "Notes on the History of Correlation," *Biometrika*, XIII (1920–21). The reader who wishes to make a thorough study of the early history of correlation should also consult the following memoirs by Galton:

"Typical Laws of Heredity," *Journal of the Anthropological Institute* (1877).

"Address to Anthropological Section of the British Association at Aberdeen," *Journal British Association* (1885).

"Regression towards Mediocrity in Hereditary Stature," *Journal of the Anthropological Institute* (1885).

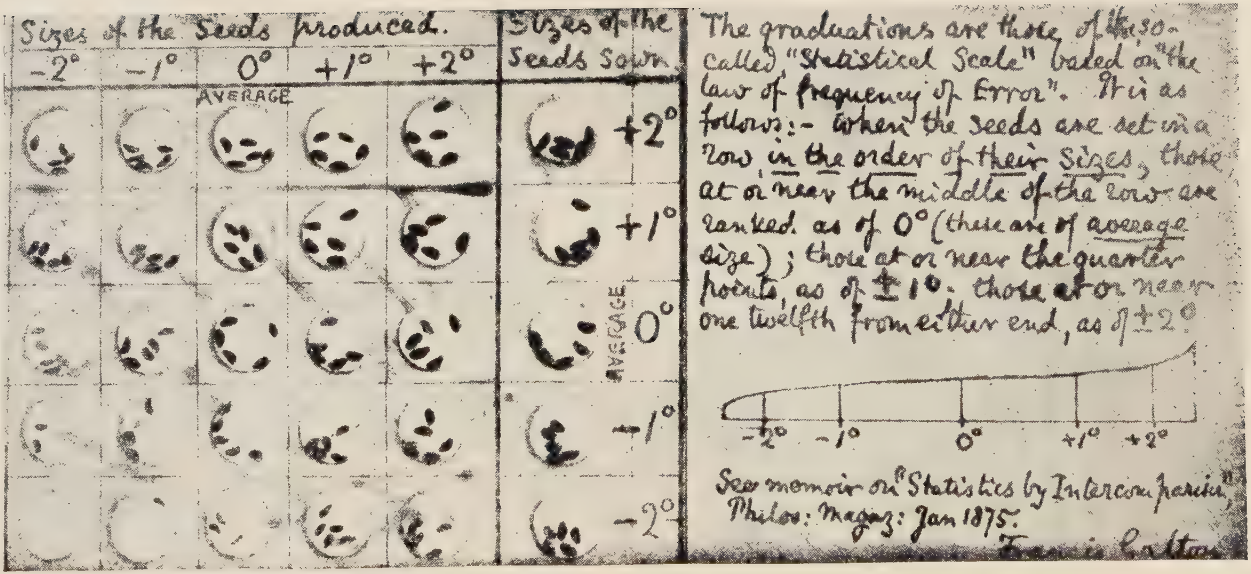
Presidential Address to Anthropological Institute (1885).

"Family Likeness in Stature," *Proceedings of the Royal Society* (1886).

"Family Likeness in Eye-Color," *Proceedings of the Royal Society* (1886).

"Pedigree Moth Breeding," *Transactions of the Entomological Society* (1887).

At the time of this writing, the first two volumes only of *The Life, Letters, and Labours of Francis Galton* have appeared. The third volume which is to deal with the statistical work of Galton is now in press, and it will undoubtedly contain much valuable material which was not accessible to the writer, and will give a more complete and authentic account of the early history of correlation than can be obtained elsewhere.



Galton's First Illustration of Correlation (about 1875)

From the *Life of Francis Galton*, by permission of Professor Karl Pearson

"In 1875 Galton was dealing with the inheritance of size in sweet-pea seeds, but before he obtained his data for sweet-peas, he appears to have tried what he could do with much smaller seeds, apparently that of cress. The correlation of the seed of mother and daughter plants was dealt with by the method of his memoir of 1875, and that is, I think, the probable date of his first crude correlation table, which he obtained from five groupings of each size of seed; the isograms are represented by ink lines on the sheet of glass covering the little compartments which contain the ranked seeds of the daughter-plants. These isograms have been smudged and almost obliterated by the wear and tear of fifty years, but can still be traced." Pearson's *Life of Francis Galton*, II, p. 392.

to demolish a multitude of dogmatic barriers by a single stroke, and to arouse a spirit of rebellion against all ancient authorities whose positive and unauthenticated statements were contradicted by modern science."[14] In the years that followed Galton made study after study on the subject of heredity, beginning with two articles on "Hereditary Talent and Character,"[15] and including *Hereditary Genius* (1869, 2d ed. 1892), *English Men of Science* (1874), *Human Faculty* (1883), *Record of Family Faculties* (1884), *Life History Album* (1884), *Natural Inheritance* (1889), and numerous lectures and memoirs on eugenics.

Birthplace of Idea of Correlation. In his *Memories*, Galton tells of the first time he recognized that regression would be the key to the problem of heredity: "As these lines are being written, the circumstances under which I first clearly grapsed the important generalisation that the laws of Heredity were solely concerned with deviations expressed in statistical units, are vividly recalled to my memory. It was in the grounds of Naworth Castle, where an invitation had been given to ramble freely. A temporary shower drove me to seek refuge in a reddish recess in the rock by the side of the pathway. There the idea flashed across me, and I forgot everything else for a moment in my great delight." (p. 300.) Pearson suggests that this "recess" deserves a commemorative tablet.

Earliest Studies. Galton's earliest studies on regression were not published, and the only available source of information is Pearson's *Life of Galton*. Here we learn that as early as 1875 Galton was conducting experiments with sweet-pea seeds to determine the law of inheritance of size. An extremely interesting cut of Galton's first crude pictorial correlation table, which Pearson thinks is of about this date, is shown here. Pearson also suggests that probably "Galton dealt with the correlation of ranks before he reached the correlation of variates, and the claim that it is a contribution of the psychologists some thirty or forty years later to the conception of correlation does not seem to me valid."

The question which was agitating Galton at this time was: "How is it possible for a whole population to remain alike in its features, as a whole, during many successive generations, if the *average* produce of each couple resemble their parents?" To get data for studying the problem, he raised sweet-peas and persuaded friends to do the same for him, he experimented

[14] *Memories*, p. 287.
[15] *Macmillan's Magazine* (1865).

with the raising of pedigreed moths, studied hounds, and finally offered prizes for records of human families.

Galton's first public statement of his discovery was made in his lecture on "Typical Laws of Heredity in Man" at the Royal Institution, in 1877. It is almost past belief that the crude data which he had been able to assemble from his sweet-peas should have suggested to him the law of regression, or *reversion*,[16] as he called it at this time; the formula for the standard error of estimate,[17] and the idea that the various arrays in the correlation table had equal variability.[18]

First Publication. For some years Galton's published works give no suggestion of the law of regression. In 1885 he delivered a presidential address before the Anthropological Section of the British Association, at a meeting held in Aberdeen. Shortly afterwards he contributed to the *Journal of the Anthropological Institute* a paper called "Regression towards Mediocrity in Hereditary Stature," in which he reviewed much of the material in the lecture. This paper contained what we would now call a correlation chart with both regression lines drawn.[19] Galton had smoothed the frequencies in his correlation chart by writing at the intersection of any two lines the mean of the frequencies in the four adjacent cells, and had then drawn contour lines through the points of equal frequency. He saw that these were roughly a set of concentric ellipses, and that sections parallel to the axes appeared to have a normal distribution. He writes in his *Memories*: "But I could not see my way to express the results of the com-

[16] "Reversion is the tendency of the ideal mean filial type to depart from the parental type, reverting to what may be roughly and perhaps fairly described as the average ancestral type. If family variability had been the only process in simple descent that affected the characteristics of a sample the dispersion of the race from its mean ideal type would indefinitely increase with the number of generations, but reversion checks this increase, and brings it to a standstill."

[17] $v = \sqrt{1-r^2}\, c_1$, or "variability of family $= \sqrt{1-r^2}$ times variability of general population."

[18] "I was certainly astonished to find the variability of the produce of the little seeds to be equal to that of the big ones; but so it was and I thankfully accept the fact, for had it been otherwise, I cannot imagine, from theoretical considerations, how the typical problem could be solved." This and the other statements concerning this lecture, are taken from Pearson's "Notes on the History of Correlation," as the original has not been available.

[19] Pearson says that this was the first correlation chart. It should be remembered, however, that in 1877 Bowditch had distributed height and weight, height and age, and weight and age in what were actually correlation charts, although he did not know how to measure correlation.

plete table in a single formula. At length, one morning, while waiting at a roadside station near Ramsgate for a train, and poring over the diagram in my notebook, it struck me that the lines of equal frequency ran in concentric ellipses. The cases were too few for my certainty, but my eye, being accustomed to such things, satisfied me that I was approaching the solution. More careful drawings strongly corroborated the first impression." (p. 302.)

"I wrote down these values, and phrasing the problem in abstract terms, disentangled from all reference to heredity, submitted it to Mr. J. D. Hamilton Dickson, Tutor of St. Peter's College, Cambridge. . . . . I asked him kindly to investigate for me the Surface of Frequency of Error that would result from these three data [the average regression of the stature of offspring from mid-parent, Q for the general population, and Q for the co-fraternity] and the various shapes and other particulars of its sections that were made by horizontal planes, inasmuch as they ought to form the ellipses of which I spoke.

"The problem may not be difficult to an accomplished mathematician, but I certainly never felt such a glow of loyalty and respect towards the sovereignty and wide sway of mathematical analysis as when his answer arrived, confirming by purely mathematical reasoning my various and laborious statistical conclusions with far more minuteness than I had dared to hope, because the data ran somewhat roughly and I had to smooth them with tender caution."[20]

Galton's Method. In studying the inheritance of stature, Galton first "transmuted" all female statures to male, multiplying them by the constant 1.08.[21] The average of the statures of the parents was then termed the "Mid-Parent;" the average of the statures of the offspring, the "Mid-Fraternity." From these he made "Tables of Stature," later called correlation tables. For each array and for the table as a whole he found the median. He also found Q[22] for the general population, for the mid-parents, and for each "Co-Fraternity," a "Co-Fraternity" being "all the adult Sons and Transmuted Daughters of a *group* of Mid-Parents who have the same stature (reckoned to the nearest inch)."[23] Examining these values

[20] This passage occurs in the Presidential Address of 1885 and also in *Natural Inheritance* (1889).

[21] This had the effect of equating means, but did not take account of the different variability in male and female statures.

[22] Galton used Q (the semi-interquartile range) as an approximate value of the probable error.

[23] Galton's co-fraternity is thus equivalent to Pearson's "type-array," except that the latter may be used in speaking of either variable.

of Q for the various co-fraternities, he found them roughly the same, regardless of whether the mid-parents were tall or short. Examining the medians he found that "However paradoxical it may appear at first sight, it is theoretically a necessary fact, and one that is clearly confirmed by observation, that the Stature of the adult offspring must on the whole be more *mediocre* than the stature of their parents."[24] As the medians of the rows and the columns proved to lie on approximately straight lines, he called the latter "lines of regression." Because Galton had used the variability—in his case Q—as the unit in which he expressed deviations, these lines of regression were symmetrically placed. In 1877 he called the slope of these lines r for *reversion.* In his paper "Regression towards Mediocrity in Hereditary Stature" (1886) he used w, and in "Correlations and their Measurement" (1888) he returned to r, which now stood for *regression.*

First Use of Term "Correlation." The term "correlation" was used in a technical sense in 1888, in Galton's paper on "Co-relations and their Measurement." Galton said there: "Co-relation or correlation of stature is a phrase much used in biology, and not least in that branch of it which refers to heredity, and the idea is even more frequently present than the phrase; but I am not aware of any previous attempt to define it clearly, to trace its mode of action in detail, or to show how to measure its degree.

"Two variable organs are said to be co-related when the variation of one is accompanied on the average by more or less variation of the other, and in the same direction. . . . . The statures of kinsmen are co-related variables; thus the stature of the father is correlated to that of the adult son, and the stature of the adult son to that of the father; the stature of the uncle to that of the adult nephew, and the stature of the adult nephew to that of the uncle, and so on; but the index of co-relation, which is what I there ["Hereditary Stature"] called 'regression,' is different in the different cases."[25]

Natural Inheritance. The concepts of regression and correlation did not become generally known until after the appearance of *Natural Inheritance* in 1889. This sums up most of Galton's work on correlation, and presents all that he contributed to the theory. It would be an excellent thing if every student beginning to study correlation could be urged to read first the introduction to *Natural Inheritance* and then the section dealing with correlation.

[24] Page 95, *Natural Inheritance.*

[25] *Proceedings of the Royal Society*, XLVII (1888).

4. EXTENSION OF THE THEORY BY PEARSON AND OTHERS

New Studies Stimulated by *Natural Inheritance*. In the same year in which *Natural Inheritance* appeared, Weldon began to measure the organs of shrimps in order to get data for a correlational study, and Pearson read a paper on the book. The latter says in his "Notes on Correlation:" "So far as I can remember what happened at all, it was as follows. I know that I was immensely excited by Galton's book of 1889—*Natural Inheritance*—and that I read a paper on it in the year of its appearance. In 1891—92 I lectured popularly on probability at Gresham College, taking skew whist contours as illustrations of correlation. In 1892 I lectured on variation, in 1893 on correlation to research students at University College, the material being afterwards published as the first four of my *Philosophical Transactions* on evolution. At this time I dealt with correlation and worked out the general theory for three, four, and ultimately n variables. The field was very wide, and I was far too excited to stop to investigate properly what other people had done. I wanted to reach new results and apply them."[26]

Correlations from Shrimps. Galton had been seriously hampered by lack of data suitable for studying correlation. To supply this, Weldon,[27] began to measure the organs of shrimps, and published the first results of his study in 1890.[28] He expresses thanks to McAlister for help with the law of error, and to Galton for general guidance: "In making this investigation, I have had the great privilege of being constantly advised and helped, in every possible way, by Mr. Galton. My ignorance of statistical methods was so great that, without Mr. Galton's constant help, given by letter at the expenditure of a very great amount of time and trouble, this paper would never have been written." Weldon followed Galton's practice of taking his deviations from the median and expressing them in terms of Q. The paper gave no suggestion of the meaning of correlation, nor did it state the numerical values of the coefficients found, but stated that certain biological principles, which he expressed in terms of correlation and variation, had been established.

[26] The last two sentences are in explanation of the reasons why he had at one time ascribed to Bravais the credit for discovering correlation, a misleading statement which has been perpetuated even until now, in spite of all Pearson's later efforts to correct it.

[27] Walter Frank Raphael Weldon (1860–1906) was one of the founders of *Biometrika*, the other two being Galton and Pearson, and a co-editor until his death. He was a biologist and a biometrist of distinction, and a professor at University College, London 1891–99, and at Oxford, 1900–06. A biography by Pearson may be found in *Biometrika*, *V* (1906), 1–52.

[28] "The Variations occurring in certain Decapod Crustacea.—I. *Crangon Vulgaris*," *Proceedings of the Royal Society*, XLVII (1890), 445–453.

Weldon's second paper on shrimps[29] appeared in 1892. In it he gave a brief description of Galton's method and its meaning and described his own method of computing r. Galton had plotted the means of the several rows (and likewise of the columns) and had fitted a straight line to them by his eye, taking the slope of this line for the value of r. Weldon's description of his own method is as follows:

"(1) In the population examined, let all those individuals be chosen in which a certain organ, A, differs from its average size by a fixed amount, Y; then, in those individuals, let the deviations of a second organ, B, from its average be measured. The various individuals will exhibit deviations of B equal to x_1, x_2, x_3, ... whose mean may be called x_m. The ratio x_m/Y will be constant for all values of Y.

"In the same way, suppose those individuals are chosen in which the organ B has a constant deviation, X, then in those individuals y_m, the mean deviation of the organ A, will have the same ratio to X, whatever may be the value of X.

"(2) The ratios x_m/Y and y_m/X are connected by an interesting relation. Let Q_a represent the probable error of distribution of the organ A about its average, and Q_b that of the organ B; then—

$$\frac{y_m/X}{x_m/Y} = \frac{Q_a^2}{Q_b^2}$$

or

$$\frac{x_m/Q_b}{Y/Q_a} = \frac{y_m/Q_a}{X/Q_b} = r, \text{ a constant.}$$

"So that by taking a fixed deviation of *either* organ, expressed in terms of its probable error, and by expressing the mean associated deviation of the second organ in terms of its probable error, a ratio may be determined, whose value becomes ± 1, when a change in either organ involves an equal change in the other, and 0 when the two organs are quite independent. This constant, therefore, measures the 'degree of correlation' between the two organs." This seems to have been the first recognition of negative correlation. Weldon plotted one regression line.

Weldon's third paper, in which he called r "Galton's function" appeared in 1893.[30]

[29] "Certain Correlated Variations in *Crangon Vulgaris*," *Proceedings of the Royal Society*, LI (1892), 2–21.

[30] "Correlated Variations in Naples and Plymouth Shore Crabs."

Speaking of the work of Weldon, Pearson writes: "So far as the present author is aware, the paper 'The Variations occurring in certain Decapod Crustacea I. *Crangon Vulgaris*' was the first to apply the methods of Galton to other zoölogical types than man. [Footnote: Galton had dealt with the weights of sweet-pea seeds, Merrifield with the sizes of moths, but they had not published fitted frequency distributions.] In this paper Weldon shows that different measurements made on several local races of shrimps give frequency distributions closely following the normal or Gaussian law. In his next paper 'On Certain correlated Variations in *Crangon Vulgaris*,' Weldon calculated the first coefficients of organic correlation, that is, the numerical measures of the degrees of interrelation between two organs or characters in the same individual. It is quite true that the complete modern methods were not adopted in either of these papers, but we have for the first time organic correlation coefficients—although not yet called by that name, tabled for four local races. These two papers are epoch-making in the history of the science, afterwards called biometry."[31]

"Correlated Averages." In this same year appeared Edgeworth's paper on "Correlated Averages."[32] This was an important paper, although almost everything in it except the use of the term "coefficient of correlation" has now been superseded. Edgeworth based his definition of correlation. upon probability, and extended the theory to three or more variables. His numerical computations involved a knowledge of the properties of determinants. Had the theory not been developed past this point, correlation would certainly never have achieved its present popularity.

First Use of Correlation to Study a School Problem. Probably the first use of correlation in a study related in any way to the work of the schools was made by an American in 1892. Experiments had been performed by Bryan on the voluntary motor ability of 789 children in Worcester, Massachusetts.[33] As a part of his larger study he wished to measure the bilateral asymmetry of the rate of the joints, and did so by a method and development which he says were due to Boas. The problem was to discover whether the rate of the right and left side joints varied together, and if not, to determine the degree of asymmetry in their development. Relieved of unfamiliar symbolism, the treatment is on this order: Let x = rate of a right side joint

[31] Biography of Weldon, *Biometrika*, V (1906).

[32] *Philosophical Magazine*, 5th series, XXXIV (1892), 190–204.

[33] "On the Development of Voluntary Motor Ability," *American Journal of Psychology*, V (1892), 123–204. See pages 163–165 for treatment of correlation.

expressed as a deviation from its mean, and y = rate of a left side joint, expressed as a deviation from its mean. Then if x and y are independent, the variation in their differences will be $\sqrt{\sigma_x^2 + \sigma_y^2}$, but if they are related the variation in the differences will not agree with this value. Let σ_d be the actual variation in these differences. Then $\frac{100\,\sigma_d}{\sqrt{\sigma_x^2 + \sigma_y^2}}$ is the relation in per cent between the chance value and the actual degree of bilateral asymmetry. The expression $\sqrt{\sigma_x^2 + \sigma_y^2}$ is obtained from the exponent in the formula for the simultaneous occurrence of two independent variables.

Pearson's First Publications on Correlation. Pearson says in his "Notes on the History of Correlation" that as early as 1893, he dealt with a number of correlation tables for long series, and demonstrated that

"(i). Smooth and definite systems of contours can arise from long series, obviously mathematical families of curves, which are (a) ovaloid, not ellipsoid, and (b) which do not possess—like the normal surface contours—more than one axis of symmetry.

"(ii). Regression curves may be smooth mathematical curves and yet not straight lines.

"(iii). If (i) and (ii) hold, then homoscedasticity is not the rule."

Pearson's first fundamental paper on correlation, "Regression, Heredity and Panmixia,"[34] was written in 1895 from his lecture notes. This paper generalized the conclusions and developed the methods of *Natural Inheritance*; showed the wide field which a purely statistical theory of heredity might be made to cover; proved that the "best value" of the correlation coefficient is $\frac{\Sigma xy}{N\sigma_x\sigma_y}$; gave the value of the probable error[35] of r as $.674509\ \frac{1 - r^2}{\sqrt{n(1 + r^2)}}$; proposed the "coefficient of variation;" suggested two formulas by which the computation of r might be more easily performed;[36] stated the general theory of correlation for three traits; and gave definitions of correlation and regression far more general than anything which had

[34] "Mathematical Contributions to the Theory of Evolution.—III. Regression, Heredity and Panmixia," *Philosophical Transactions*, A, CLXXXVII (1896), 253–318. An abstract of the paper may be found in the *Proceedings of the Royal Society*, LIX (1896) 69–71.

[35] In 1898 he showed that this formula was valid only under restricted conditions See page 76.

[36] These formulas are similar to, though not identical with those used by Toops (1921) and Otis (1923) in their correlation charts. See also page 129.

preceded. The work of this paper proceeds on the assumption that variation in any organ follows the normal law, but Pearson definitely suggests the need for studying skew variations and skew correlation, and in a footnote says: "Mr. G. U. Yule points out to me that if the coefficient of regression be constant for the arrays of all types, then it follows that *whatever be the law of frequency*, the coefficient of regression must equal $r\sigma_1/\sigma_2$, where $r = S(xy)/(n\sigma_1\sigma_2)$" (p. 268).

5. PARTIAL AND MULTIPLE CORRELATION

Normal Correlation for Three Variables. As has been stated, Plana, Laplace, Gauss, Bravais, Czuber, De Forest, and perhaps others, dealt with the probability of the simultaneous occurrence of errors in three or more related variables, all of which were subject to the normal law of distribution, but none of them attempted to measure the strength of that relationship. Edgeworth's treatment of the problem also proceeded from the probability function, and lead to a very difficult method of computation. It was Pearson who first succeeded in expressing what we now call multiple and partial correlation coefficients in terms of the coefficients of zero order, and thus brought the computations within the power of those who are not mathematicians. He also used the normal probability function as his point of departure. In "Regression, Heredity and Panmixia," he gave the partial regression equation in which h_1 is predicted from combined values of h_2 and h_3:

$$h_1 = \frac{r_3 - r_1 r_2}{1 - r_1^2} \cdot \frac{\sigma_1}{\sigma_2} h_2 + \frac{r_2 - r_1 r_3}{1 - r_1^2} \frac{\sigma_1}{\sigma_3} h_3.$$

The coefficients of h_2 and h_3 he called "coefficients of double regression." He also gave at this time the formula for the standard error of h_1 when predicted by this equation.

Work of Yule. The next year Yule—who was then a student of Pearson's—published two papers[37] in which he made the theory general for surfaces of correlation other than the normal. Yule assumed rectilinearity of regression and introduced the use of "characteristic equations" such as $x_1 = b_{12}x_2 + b_{13}x_3$. The parameters b_{12} and b_{13} he obtained by the method

[37] "On the Significance of Bravais' Formulae for Regression, in the case of Skew Correlation," *Proceedings of the Royal Society*, LX (1897); 477–489.

"On the Theory of Correlation," *Journal of the Royal Statistical Society*, LX (1897), 812–854.

of least squares. He found that the values thus derived were the same as those that Pearson had obtained in the case of normal correlation. The standard error of estimate of x_1 when x_2 and x_3 are known, Yule[38] wrote as $\sigma_1 \sqrt{1 - R_1^2}$, and said that "$R_1$ may in fact, be regarded as a coefficient of correlation between x_1 and (x_2x_3)." This use of R for the coefficient of multiple correlation has survived to the present, although there is now a diversity of use as to the manner of writing its subscripts and their position. Yule at this time used the term "net coefficient of correlation" for what Pearson later called "partial correlation."

6. THEOREMS GROWING OUT OF THE CORRELATION OF SUMS

Origin of the Formulas. A group of formulas of considerable practical importance in connection with educational tests have their common origin in the correlation of sums. Among these are the Spearman-Brown "prophecy formula," the index of reliability, and much of the recent work of Kelley on formulas relative to the "true score" on a test. The algebraic derivation of these is a natural outgrowth of the fundamental work on probable errors developed by Pearson, Sheppard, and Filon.[39] The general method of approach and the underlying theory is implicit in papers by these authors published about 1898, but the applications to educational and psychological tests were made somewhat later. In 1904 Spearman called attention to the way in which inaccuracies in test scores reduce or attenuate correlation, and in 1910 he introduced the term "reliability coefficient," which he defined as "the coefficient between one half and the other half of several measurements of the same thing."

The general matrix from which all such formulas come is the formula for the correlation of x and y when x is determined by p and y by q independent measurements, it being assumed that any one series of measurements of a variable yields the same mean and the same standard deviation as any other series.

Let $\bar{r}_{xy}$ = the mean of all the pq different correlations which can be obtained between one series of measurements of x and one series of measurements of y,

[38] Pearson had written this in the form

$$\Sigma_1 = \sigma_1 \sqrt{\frac{1 - r_1^2 - r_2^2 - r_3^2 + 2\, r_1\, r_2\, r_3}{1 - r_1^2}}$$

[39] Pearson and Filon, "On the Probable Errors of Frequency Constants and on the Influence of Random Selection on Variation and Correlation," *Philosophical Transactions*, A, CXCI (1897–98), Part I, 229–311.

Sheppard, "On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation," *Philosophical Transactions*, A, CXCII (1899), 101–169.

and $\bar{r}_{x_1x_2}$ = the mean of all the correlations which can be obtained between two different series of measures of x,

and $\bar{r}_{y_1y_2}$ = the mean of all the correlations which can be obtained between two different series of measures of y.

$$\text{Then } r_{(x_1+x_2+\cdots+x_p)\,(y_1+y_2+\cdots+y_q)} = \frac{pq\,\bar{r}_{xy}}{\sqrt{p+p(p-1)\bar{r}_{x_1x_2}}\,\sqrt{q+q(q-1)\bar{r}_{y_1y_2}}}.$$

When $p = q =$ infinity,[40] this reduces to the correction for attenuation. When $p = 1$ and q is infinite, it yields the correlation between an observed score in x and a "true score" in y, that is $\frac{\bar{r}_{xy}}{\sqrt{\bar{r}_{y_1y_2}}}$. When $p = q$ and x and y are measurements of the same variable, this becomes the Spearman-Brown "prophecy formula." Closely as these variants are related in theory, they were not announced simultaneously.

In general the formulas in this group are used in connection with an attempt to minimize or at least to measure the untoward effect of errors of observation upon statistical work, particularly of those errors which are caused by unreliability in testing instruments; or to predict mathematically the theoretical statistical advantage which might be gained by a stated amount of improvement in the reliability of the tests.

The Standard Deviation of Sums and Differences. The laws concerning the variability of the sums and differences of *independent* variables are very old. In 1809 Gauss stated the rule for the variability of the mean of a set

[40] Dividing both terms of the fraction by pq, we get

$$\frac{\bar{r}_{xy}}{\sqrt{\frac{1}{p}+\frac{p-1}{p}\bar{r}_{x_1x_2}}\,\sqrt{\frac{1}{q}+\frac{q-1}{q}\bar{r}_{y_1y_2}}} = \frac{\bar{r}_{xy}}{\sqrt{\frac{1}{p}+\bar{r}_{x_1x_2}-\frac{1}{p}\bar{r}_{x_1x_2}}\,\sqrt{\frac{1}{q}+\bar{r}_{y_1y_2}-\frac{1}{q}\bar{r}_{y_1y_2}}}$$

To see how the special formulas are derived from this, we have only to notice what happens when (1) $p = 1$, (2) p is very large, or (3) $p = 1$ and $x = y$.

(1) Whenever $p = 1$, the first radical is equal to $\sqrt{1} = 1$. A similar statement applies to the second radical whenever $q = 1$.

(2) Whenever p becomes infinitely large, $1/p = 0$. Then the first radical reduces to $\sqrt{r_{x_1x_2}}$. Whenever q becomes infinitely large, the second radical reduces to $\sqrt{r_{y_1y_2}}$.

(3) Whenever $p = q$ and x and y are measurements of the same variable, the two radicals are identical, and their product may be written as either $p + p(p - 1)\,r_{x_1x_2}$ or $q + q(q - 1)\,r_{y_1y_2}$.

Combinations of these conditions produce certain of the special formulas discussed in this section.

of independent observations. Among the many practical rules for statistical computations found in Encke's treatise on least squares (1832), are three which, with some changes in symbolism, are as follows:

(1) If $X_0 = X_1 + X_2$, then $p.e._0 = \sqrt{(p.e._1)^2 + (p.e._2)^2}$

(2) If $X_0 = aX_1 + bX_2 + cX_3 + \ldots\ldots$,

then $p.e._0 = \sqrt{a^2(p.e._1)^2 + b^2(p.e._2)^2 + c^2(p.e._3)^2 + \ldots\ldots}$

(3) If $a_1, a_2, \ldots\ldots a_n$ are independent variables whose probable errors are respectively $p.e._1, p.e._2, \ldots\ldots p.e._n$, the most probable arithmetic mean of the variables is obtained when these are weighted in inverse proportion to the squares of their probable errors, that is

$$X = \frac{\dfrac{a_1}{(p.e._1)^2} + \dfrac{a_2}{(p.e._2)^2} + \ldots\ldots + \dfrac{a_n}{(p.e._n)^2}}{\dfrac{1}{(p.e._1)^2} + \dfrac{1}{(p.e._2)^2} + \ldots\ldots + \dfrac{1}{(p.e._n)^2}}$$

All three of these are used in current educational practice, the first one quite extensively. The last, which was proved again by Kelley,[41] is obviously a symbolic statement of one method of combining the results of test scores which have unequal reliability.

Presumably these rules have been known to astronomers and other scientists since Encke's time. Airy utilized them in his *Theory of Errors of Observations*, saying, "Hence we have this very remarkable result. When two fallible determinations X and Y are added algebraically to form a result Z, the law of frequency of error for Z will be the same as for X and Y, but the modulus will be formed by the theorem, square of modulus for Z = square of modulus for X + square of modulus for Y. . . . . It cannot be too strongly enforced on the student that the measures which determine X must be absolutely and entirely independent of those which determine Y." He added similar rules for the probable error, the middle error, and the mean square error of $X + Y$, of $X - Y$, of the unweighted sum of any number of variables, and of the weighted sum of any number of variables. Furthermore there is an interesting chapter on "Entangled Measures" indicating that Airy knew that the rule $\sigma_{x-y} = \sqrt{\sigma_x^2 + \sigma_y^2}$ does not hold unless X and Y are uncorrelated. Since the coefficient of correlation

[41] *Statistical Method*, p. 325 formula [309]. In a letter to the writer (December 7, 1926) Professor Kelley says, "This formula is certainly not original with me, as I had seen reference to it before 1923. However, a long search on my part failed to reveal any proof of it so that the proof here given may be the first of its kind. Should you find an earlier proof [309] I would much like to be told of it."

had not been discovered, he could not write the full formula. Here was another unappropriated opportunity for the discovery of the coefficient of correlation.[42]

In his paper on the theory of error applied to cases of normal frequency (1899), Sheppard had a section on the mean square and mean product of composite numbers, that is to say, of sums, and also considered the special case of the mean square of the sum of two measures. Obviously the standard deviation of a sum follows from its mean square by the mere process of dividing by the number of cases and extracting the square root. Just when the exact formula $\sigma_{x-y} = \sqrt{\sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y}$ first found its way into print, the writer does not know. There is a discussion of this and of some related theorems in the first edition of Yule's *Introduction to the Theory of Statistics* (1911) and the shorter formula is mentioned as an important special case of the longer one. This seems a rather late date for the first publication of a formula so closely related to a portion of statistical theory enunciated in one of the fundamental papers on the law of errors, and it is not unlikely that an earlier use has escaped notice.

In current educational practice, this formula is much abused,[43] the shorter form being used repeatedly where only the longer is correct.

The Correlation of Sums and Differences. The coefficient of correlation between two sums follows from Sheppard's mean squares and mean products of composite numbers by mere division. In the paper mentioned, however, Sheppard did not make that division and announce the formula for the correlation of sums. In the text by Yule mentioned above, the chapter on "Miscellaneous Theorems Involving the Use of the Correlation-Coefficient" is suggestive of the method of correlating sums, although it does not develop the general theorem. A list of exercises at the end of the chapter contains the problem to find the correlation between $X_1 + X_2$ and $X_2 + X_3$ when X_1, X_2, and X_3 are uncorrelated; to find the correlation between X_1, and $aX_1 + bX_2$ when X_1 and X_2 are uncorrelated; and to find the correlations between x_1, x_2, and x_3 when the relation

$$ax_1 + bx_2 + cx_3 = 0$$

holds for all values of the variables.

Certain special aspects of the topic were treated by Spearman in 1910, and three years later, in a paper entitled "Correlations of sums or differ-

[42] Cf. page 92.

[43] Walker, "Note on the Standard Deviation of a Difference" *Journal of Educational Psychology*, XX (1929), 53–60.

ences,"[44] he gave a general treatment, with the comment "In fact, all the main formulas more or less elaborately demonstrated in that paper[45] turn out to be immediate corollaries of our simply obtained (1).[46] Hence it was a mistake in that paper to assume them at all independent of one another. Many results of other writers can be got with equal facility. For instance a few months ago, Woodworth proved[47] that the correlation of $(x + y)$ with x is $1/\sqrt{2}$. . . . Further instances have appeared above in the values that had been previously reached in various manners by Udny Yule."

The Prophecy Formula. This well-known formula was published in 1910, when it appeared simultaneously in a paper by Spearman[48] and in one by Brown[49] in the same number of the *British Journal of Psychology*. Spearman's work deduces the formula in question as a particular case of a more general theory. Brown gives first the very special case of $n = 2$ and generalizes this to obtain the prophecy formula, which is itself a special case of the Spearman formula. Spearman's approach is as follows: "It is often very useful to be able to estimate how much this reliability coefficient will probably be increased by any given additional number of measurements, or how much it will probably be reduced by any given diminution in the number of measurements. It can be shown that the following relation holds good:

$$r_{x[p],\,x[p]} = \frac{p \cdot r_{x[q],\,x[q]}}{q + (p - q) \cdot r_{x[q],\,x[q]}},$$

where $r_{x[q],\,x[q]}$ is the known reliability coefficient[50] of x when the latter has been measured $2q \times i$ times, i being any number, and $r_{x[p],\,x[p]}$ is the required most probable reliability coefficient if x be measured $2p \times i$ times." In an appendix he has the form

[44] *British Journal of Psychology*, V (1912–13), 417–426.

[45] "Correlation calculated from Faulty Data," (1910).

[46] A formula equivalent to the one on page 113.

[47] "Combining the Results of Several Tests: A Study in Statistical Method," *Psychological Review*, XIX (1912), 97–123.

[48] "Correlation calculated from Faulty Data," British *Journal of Psychology*, III (1910), 71–295. The formula is on pages 290–291.

[49] "Some Experimental Results in the Correlation of Mental Abilities," *British Journal of Psychology*, III (1910), 296–322, republished as Chapter III in *Essentials of Mental Measurement* (1911). The formula is on page 299 of the paper, page 102 of the book.

[50] Thus $r_{x[3],\,x[3]}$ would mean the reliability coefficient obtained by correlating the score on a pool of three test forms with the score on a pool of three other comparable forms, six forms in all being used.

$$r_{x[p],\, x[p]} = \frac{p \cdot r_{x[1],\, x[1]}}{1 + (p - 1)\, r_{x[1],\, x[1]}}.$$

Five pages later in the same journal, Brown published a table of test results, one column of which was headed "Rel. coeff. (r_2) for amalgamated pair of tests, $r_2 = \frac{2\, r_1}{1 + r_1}$." In a footnote is the following explanation:

"r_2 measures the extent to which the amalgamated results of the two tests would correlate with a similar amalgamated series of two other applications of the same test. If x_1, x_2, x_1', x_2', be two pairs of results (x denoting, as usual, deviations from the mean value) we may assume that

$$\sigma_{x_1} = \sigma_{x_2} = \sigma_{x_1'} = \sigma_{x_2'} = \sigma_x \text{ (say)},$$

and that $S(x_1x_1') = S(x_1x_2') = S(x_2x_1') = S(x_2x_2') = n\, \sigma_x^2\, r_1$.

Hence we get $r_2 = \frac{2\, r_1}{1 + r_1}$.

It is easily seen that the amalgamation of 4 tests gives a reliability coefficient $= \frac{4\, r_1}{1 + 3\, r_1}$; and, in general, for n tests we have

$$r_n = \frac{n\, r_1}{1 + (n - 1)\, r_1}.$$

The last formula furnishes a ready means of determining from the reliability coefficient of a single test, the number of applications which would be necessary to give an amalgamated result of any desired degree of reliability."

The Index of Reliability. Using Spearman's general formula just quoted, Abelson[51] let $p = 1$ and $q =$ infinity in order to find the probable correlation of a score on a single test with the average score on an infinite number of different tests or estimates of that function, which he termed true values, and found this to be the square root of the reliability coefficient. He did not state an algebraic formula, but merely the arithmetic computation applicable to the problem in hand. The treatment is probably due directly to Spearman. A first appendix to the main paper is marked "Kindly supplied by Prof. Spearman," and the acknowledgment may be intended to apply also to the second appendix in which this analysis is found.

[51] "The Measurement of Mental Ability of 'Backward Children,'" *British Journal of Psychology*, IV (1911), 268–314.

Kelley made an independent derivation[52] of the same formula in 1916, being at that time unaware of the work of Spearman on the correlation of sums and differences.[53]

The name *index of reliability* appears to be more or less accidental. In his discussion of the uses of the measure, Kelley wrote, " . . . . the extent to which the grade determined by means of this test . . . . would correlate with the true spelling ability of the individual is probably an even more significant index of reliability." When Monroe published the formula in his *Introduction to the Theory of Educational Measurements* (1923) he capitalized the phrase, ascribed it to Kelley, and established *Index of Reliability* as a definite term.

Related Theorems. Educational research is now making considerable use of a set of formulas derived from the index of reliability and from the general concepts of correlation, regression, and standard error of estimate. Among these may be mentioned the standard error of a score, used by Kelley since 1916, the standard error of a true score predicted from a single score, the regression equation for predicting a true score from a single score, effect of range upon reliability ceofficients, and the like. These belong to that part of educational statistical theory which is still developing, and therefore will not be treated here in detail. Most of these theorems have been developed by Kelley or applied to education by him, and his discussions are so accessible as to justify no repetition here. In this connection the reader is particularly referred to the following of his papers and books: "The Reliability of Test Scores," *Journal of Educational Research*, (1921); *Statistical Method*, p. 212–233; and *Interpretation of Educational Measurements* (1927), pages 171–181. Several recent papers by Kelley, Holzinger and others are mentioned in the bibliography at the end of this book.

7. CORRECTION FOR ATTENUATION

Proposed by Spearman. The need of a method for correcting raw coefficients of correlation to compensate for the attenuating effect of errors in measurement was first pointed out by Spearman in 1904, in a paper which had considerable influence upon educational and psychological research in

[52] "A Simplified Method of Using Scaled Data for Purposes of Testing," *School and Society*, IV (1916), 34–37, 71–75. This formula is on page 74.

[53] This was indicated in a letter to the writer, December 7, 1926. See also a comment on page 498 of the *Journal of Educational Psychology*, XIV (1923); "Let me at this time further express my regret that when I derived formulas for the standard deviation of sums and for the correlation between sums I was unfamiliar with Spearman's work and therefore did not credit him with the formulas which I gave."

America.[54] This paper contained the germs of a number of other important topics. It was followed some months later by a second[55] in which Spearman made use of two of the formulas suggested in his first paper for eliminating the effect of unreliability in the measuring instruments.

In the first of these articles, Spearman offered without proof the formula

$$r_{pq} = \frac{r_{p'q'}}{\sqrt{r_{p'p'}\, r_{q'q'}}}$$

where $r_{p'q'}$ = the mean of the correlations between each series of values obtained for p with each series obtained for q,

$r_{p'p'}$ = the mean of all possible correlations between the different series of independently obtained values of p,

$r_{q'q'}$ = the same as regards q,

and r_{pq} = the required real correlation between the true values of p and q.

The value of $r_{p'q'}$ might be obtained he said, by taking either the arithmetic or the geometric mean of the several correlations between p and q, and both formulas are presented. While offering no proof for either form Spearman implied that he had a mathematical derivation which he would publish later. Another formula, empirically arrived at, is given in the same paper as providing a check upon the first one. It is

$$r_{pq} = \frac{\sqrt[4]{mn}\, r_{p''q''} - r_{p'q'}}{\sqrt[4]{mn} - 1},$$

where m and n are the number of independent ratings for p and q respectively, and $r_{p''q''}$ is the correlation of the amalgamated series for p with the amalgamated series for q.

The numerical illustrations adduced in this paper are somewhat startling. A raw coefficient of 0.38 is raised by the first method to 1.07 and by the second to .97 or to 1.05 for different values of m and n. All of these Spearman characterizes as "approximately 1," adding that "the correspondence advanced from 0.38 to absolute completeness."[56]

Almost immediately the proposed formula drew a sharp criticism from Pearson, and the suggestion that "Perhaps the best thing at present would

[54] "The Proof and Measurement of Association between Two Things," *American Journal of Psychology*, XV (1904), 72–101.

[55] " 'General Intelligence' Objectively Determined and Measured," *American Journal of Psychology*, XV (1904), 201–292.

be for Mr. Spearman to write a paper giving algebraic proofs of all the formulas he has used, and if he did not discover their erroneous character in the process, he would at least provide tangible material for definite criticism, which it is difficult to apply to mere unproven assertions."[57]

In 1907 Spearman published a proof[58] of the formula

$$r_{pq} = \frac{\sqrt[4]{r_{p_1 q_1} \cdot r_{p_1 q_2} \cdot r_{p_2 q_1} \cdot r_{p_2 q_2}}}{\sqrt{r_{p_1 p_2} \cdot r_{q_1 q_2}}},$$

on the assumption that "fluctuations in the two series of measurements are independent of each other," saying that in practice it would usually be satisfactory to assume that the two series of measurements of the same series had been conducted with equal accuracy and that the arithmetic might be substituted for the geometric mean in the numerator. At this time Spearman does not seem to have laid great emphasis upon the practical necessity of conforming to the assumption that errors of measurement should be uncorrelated, but used it primarily for convenience in proof.

Proof by Yule. The following year, in a private letter to Spearman, Yule suggested a simple proof of the formula, with a more explicit statement of the assumptions involved.[59] These assumptions are that errors of measurement are not only uncorrelated with each other, but also are uncorrelated with the traits measured.

Empirical Test by Thorndike. In connection with his *Measurement of Twins* (1905), Thorndike checked the last of the formulas enunciated by Spearman in 1904, and found the result by the correction formula slightly higher than the result obtained empirically from more accurate data. In 1907 he took a set of accurate measurements and made them artificially erroneous.[60] Using this specially created data to test the correction for

[56] See page 122.

[57] "Addenda, April, 1904," *Biometrika*, III, p. 160.

[58] "Demonstration of Formulae for True Measurement of Correlation," *American Journal of Psychology*, XVIII (1907).

[59] The proof appears in a mathematical appendix to an article by Spearman, "Correlation Based on Faulty Data," *British Journal of Psychology*, III (1910), 271–295. For this proof see pages 294–295. A somewhat abbreviated form of the same proof is to be found in Yule's *Introduction to the Theory of Statistics*, London, 5th ed., 1919, p. 213; in Brown's *Essentials of Mental Measurement*, 1911, and in *Essentials of Mental Measurement*, 1921, by Brown and Thomson.

[60] *Empirical Studies in the Theory of Measurement, Archives III*, 1907. See in particular pages 35–41.

attenuation, he found a fairly satisfactory correspondence between the corrected coefficient derived from the faulty data and the uncorrected coefficient derived from accurate measures.

Alternate Form Suggested by Boas. Boas suggested the alternate form

$$r_{pq} = r_{p'q'} \frac{\sigma_p \sigma_q}{\sqrt{\sigma^2_{p'} - \sigma^2_{e\,p'}}\sqrt{\sigma^2_{q'} - \sigma^2_{e\,q'}}},$$

but it seems to have attracted little attention.[61]

Criticism by Brown. Soon Brown brought out several papers[62] in which he criticized the correction formula, stating the hypothesis from which the formula was derived and suggesting two relatively simple methods of testing whether in a particular case that hypothesis is justified. The criteria which he proposed are:

(1) $\Sigma x_1 y_1 - \Sigma x_2 y_2 = 0$ within the limits of the probable error of that difference,

(2) $r_{(x_1 - x_2)\,(y_1 - y_2)} = 0$ within the limits of the probable error of r. These are necessary but not sufficient conditions for the applicability of the formula. Brown cited two experiments in which his data failed to conform to these conditions. The whole subject was reviewed once more by Brown and Thomson in *Essentials of Mental Measurement* (1921) in Chapter VIII, where they expressed their judgment that the fundamental assumptions are almost never fulfilled in practice.

Further Work by Spearman. Stimulated by the criticisms[63] which had been showered upon his correction for attenuation, Spearman returned to the subject in 1910, in his paper on "Correlation Calculated from Faulty Data."[64] It should be cause for regret to American educators that this

[61] See Thorndike, *Empirical Studies*, p. 36.

[62] "Some Experimental Results in Correlation," *Comptes Rendus du VI^{me} Congres International de Psychologie*, Geneva, August, 1909.

Footnote at end of "An Objective Study of Mathematical Intelligence," *Biometrika*, VII (1910), p. 361.

"Some Experimental Results in the Correlation of Mental Abilities," *British Journal of Psychology*, III (1910), 296–322. See especially pages 318–322.

Essential of Mental Measurement, 1911, p. 82 *et seq.*

[63] In addition to those already mentioned, see also Wissler, "The Spearman Correlation Formula," *Science*, XXII (1905), 309.

[64] *British Journal of Psychology*, III, pp. 271–295.

paper was published in England and has probably not been so well known in this country as the two earlier and less fundamental discussions. Here the assumptions underlying the correction were clearly stated, their practical implications discussed, and the mathematical theory developed in an appendix.

Probable Error of the Corrected Coefficient. In this paper of 1910, Spearman, with some assistance from Filon to whom he credits part of the proof, worked out a formula for the probable error of the corrected r.[65] Later Kelley made an independent derivation of the same probable error, getting a result considerably smaller than Spearman's. The difference in their results appears to have arisen from confusion in the subscripts of one of the r's in Spearman's first statement of the problem.[66] The importance of this reduction in the supposed size of the probable error of the corrected r has been stated by Kelley: "With probable errors available there is no excuse for the indiscriminate averaging of corrected coefficients having values above and below 1.00, yielding possibly an average nearly equal to one. If we have a corrected coefficient equal to .90 with probable error of .02, and a second equal to 1.10 with a probable error of .02, we may conclude that neither coefficient is a chance variation from 1.00, and further that the fundamental hypotheses of similar tests, lack of correlation between errors, etc., underlying the idea of a reliability coefficient, must be absent in the case of the data yielding the corrected coefficient 1.10. A corrected coefficient greater than 1.00 is just as absurd as a 'raw' coefficient greater than 1.00, and if positively found, as for example, $1.10 \pm .02$, it demands a reëxamining of hypotheses as truly as would the latter were it found to be greater than 1.00. Only in case corrected coefficients differ from 1.00 by such small amounts that the value 1.00 is well within the likelihood of occurrence, judged by the probable errors of the corrected coefficients, is it sound to average several such corrected coefficients to secure a measure of general tendency."

Popularity of Correction Formula. The prominent place which this formula has attained in educational statistical practice appears to be due to several causes. In the first place, there is the ever-recurring practical need of some method of off-setting the imperfections of the measuring instruments in the field of tests and also the impressive fashion in which this formula may raise an originally low correlation to a larger value. In the

[65] P. 292–294.

[66] See Kelley's *Statistical Method* (1923), pp. 208–212.

second place there is the chronological accident that the first statement of the correction appeared in print while Thorndike was preparing the first edition of *Mental and Social Measurements*, very shortly before its publication. In that brief period there was no time for experimental verification or for critical study of the underlying assumptions, which, it should be noted, had not been stated by Spearman. Thorndike presented the formulas as given by Spearman with a statement of what the latter had said concerning them, and the prestige of their two names caused many psychologists to accept the formulas at once and almost without reservation. In the third place, to understand the reasons why Spearman's formula has so often been uncritically used by educators, has been so often accepted on faith, it must be borne in mind that the appearance of these two papers of Spearman's in 1904 was a powerful factor in making a "correlational psychology" popular in America.

We have never had in English a single word for this process of correcting a coefficient of correlation for attenuation, although in German it has the admirable and suggestive title of "Die Ergänzungsformel."[67]

8. INDEX CORRELATIONS

The misleading character of a coefficient of correlation calculated between two indices[68] was pointed out by Pearson in 1897 in a paper[69] in which he gave the mathematical formulas for the mean of an index, the standard deviation of an index, and the coefficient of correlation between two indices, in terms of the means, coefficients of variation, and coefficients of correlation of the four simple variables involved in the ratios. He defined spurious correlation of this type as "the correlation which will be found between indices, when the absolute values of the organs have been selected purely at random," suggested that in finding correlation between the indices the amount of this spurious correlation ought always be stated, and indicated several applications to biology. As an illustration, he postulated one thousand skeletons obtained by distributing component bones at random. "Between none of their bones will these individuals exhibit correlation. Wire the skeletons together and photograph them all, so that their stature in the photographs is the same; the series of photographs, if measured, will

[67] Krueger and Spearman, "Die Korrelation zwischen verschiedenen geistigen Leistungsfähigkeiten," *Zeitschrift für Psychologie*, 44, Part I, 1907.

[68] That is, fractions, as for example, educational or intelligence quotients.

[69] "Mathematical Contributions to the Theory of Evolution. . . . . On a Form of Spurious Correlation which may arise when Indices are used in the Measurement of Organs," *Proceedings of the Royal Society*, LX (1896–97), 489–498.

show correlation between their parts." To this paper Galton contributed an appendix[70] with explanatory comments and a diagram.

Some years later Pearson suggested a method for circumventing the use of ratios in the correlation of death rates,[71] and Yule published a paper[72] saying that the process of correction was unnecessary in this situation. About the same time Pearson published in *Biometrika* another study[73] on the problem in which he included certain formulas worked out by M. Greenwood, Jr., and illustrations based upon correlations made by Alice Lee, Julia Bell, and Amy Barrington.

A further "Study of Index Correlations,"[74] was made in 1914 by J. W. Brown, M. Greenwood, Jr., and Frances Wood, in which the authors investigated the relations between the partial correlation $r_{xy \cdot z}$ and the partial correlation between the ratios $\frac{x}{z}$ and $\frac{y}{z}$ with z held constant. They utilized both algebraic methods and extensive computations of correlation coefficients from data relating chiefly to birth rates and death rates.

Neifeld has recently discussed[75] the different correlation coefficients which can be formed among ratios composed of x_1, x_2, x_3, and x_4, according as x_4 appears in the same ratio with x_1, x_2, or x_3, and the relations among these coefficients, and he has worked out index correlations for concrete data from economics, illustrating the method of partial correlation applied to indices.

Holzinger made a study[76] of the validity and reliability of an index and of the relation between the simple variables which would increase the validity and the reliability.

Thomson and Pintner[77] compared correlations based on intelligence

[70] "Note to the Memoir by Professor Karl Pearson, F.R.S., on Spurious Correlation," *Proceedings of the Royal Society*, LX (1896–97), 498–502.

[71] Pearson, Karl, Lee, Alice, and Elderton, Ethel M., "On the Correlation of Death Rates," *Journal of the Royal Statistical Society*, LXXIII (1910), 534–539.

[72] "Interpretation of Correlations between Indices or Ratios," *Journal of the Royal Statistical Society*, LXXIII (1910), 644–647.

[73] "On the Constants of Index-Distributions as deduced from the like Constants for the Components of the Ratios, with Special Reference to the Opsonic Index," *Biometrika*, VII (1909–10), 531–541.

[74] *Journal of the Royal Statistical Society*, LXXVII (1913–14), 317–346. This paper came from the Statistical Department of the Lister Institute of Preventive Medicine.

[75] Neifeld, M. R., "A Study of Spurious Correlation," *Journal of the American Statistical Association*, XXII (1927–28), 331–338.

[76] "Formulas for the Correlation between Ratios," *Journal of Educational Psychology*, XIV (1923), 344–346.

[77] "Spurious Correlation and Relationship between Tests," *Journal of Educational Psychology*, XV (1924), 433–444.

quotients with those based on mental ages and showed, by both artificial and real examples, that either correlation might be spuriously high. They showed that when chronological age is held constant, the correlation between mental ages would be the same as the correlation between intelligence quotients and recommended this procedure. They also gave as the criterion for an unselected group, that intelligence quotient and chronological age should be uncorrelated, which is equivalent to the statement that

$$r_{(MA)(CA)} = \frac{\sigma_{CA}}{\text{Mean}_{CA}} \bigg/ \frac{\sigma_{MA}}{\text{Mean}_{MA}}.$$

Recent educational research has produced many investigations in which the size of an index correlation is of central significance. A survey[78] of the use of such correlations in recent educational literature, with an analytical and graphical study of the relationships which may cause ratio correlations to be large or small, positive or negative, has been made by Dorothy Rowell.

9. MEASURES OF RELATIONSHIP OTHER THAN THE PEARSON PRODUCT-MOMENT

Correlation between Attributes. The first attempts to define the degree of correspondence between two attributes which are susceptible of qualitative description but not of quantitative measurement grew out of the study of formal logic rather than out of statistical research. The development of measures of association and contingency among attributes might well be considered as a late chapter in the history of symbolic logic. The remote beginnings might possibly be traced back to Aristotle, and some slight contributions possibly ascribed to many of the long line of logicians of the last two thousand years. The more immediate sources of a "calculus of deductive reasoning," however, are to be found about the middle of the 19th century. In 1843 John Stuart Mill[79] wrote: "Whenever the nature of the subject permits the reasoning process to be without danger carried on mechanically, the language should be constructed on as mechanical principles as possible; while in the contrary case it should be so constructed that there shall be the greatest possible obstacle to a mere mechanical use of it.[80]

[78] As yet unpublished.

[79] John Stuart Mill (1806–1873), English philosopher and economist.

[80] *System of Logic, Ratiocinative and Inductive, being a Connected View of the Principles of Evidence and the Methods of Scientific Investigation*, London, 1843, later editions in

Augustus De Morgan showed that definite numbers might be made the basis of syllogistic argument, as well as the "all," "a part," and "none" of previous logicians, and that from such premises might be deduced numerical limits of logical classes.[81] About the same time, Boole[82] undertook to reduce logic to a form of algebra,[83] in which a logical proposition is expressed as an equation. This highly original piece of work was followed a few years later by his *Laws of Thought*,[84] where in the chapter "Of Statistical Conditions"[85] he develops the upper and lower limits of the number of individuals comprised in logically defined classes, from the number of individuals in logically related classes.[86] In all his work Boole displays a remarkable command of symbolic methods and perfect confidence in the meaning of the results obtained by operations upon abstract symbols.

Boole's work in turn stimulated Jevons,[87] who though disagreeing with Boole in several important particulars,[88] acknowledges the former's work as the foundation upon which his own contributions rested. By 1870,

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1846, 1860, 1862, 1872, 1879, 1900, 1906, and other years not ascertained, there being ten editions to date.

[81] *Formal Logic, or the Calculus of Inference, Necessary and Probable*, London, 1847, reprinted under the editorship of A. E. Taylor, London, 1926. See in particular Chapter VIII, "On the numerically definite Syllogism."

[82] George Boole (1815–1864), an English logician and mathematician.

[83] *The Mathematical Analysis of Logic*, 1847.

[84] *An Investigation of the Laws of Thought on which was Founded the Mathematical Theories of Logic and Probabilities*, London, 1854. This was reprinted in *George Boole's Collected Logical Works*, 2 vol., Chicago, 1916.

[85] Chapter XIX, "By the term statistical conditions, I mean those conditions which must connect the numerical data of a problem in order that those data may be consistent with each other and therefore such as statistical observations might actually have furnished." (Page 295 in the original, page 311 in Vol. II of the reprint.)

[86] In the preface to his *Mathematical Analysis of Logic*, Boole suggests that his attention had been attracted to this subject by the acrimonious controversy then raging between De Morgan and Sir William Hamilton. The appendix to De Morgan's *Formal Logic* contains an account of this unpleasant exchange, as does also Mrs. De Morgan's biography of her husband.

[87] William Stanley Jevons (1835–1882), F.R.S., was a professor of logic and mental and moral philosophy and Cobden lecturer in political economy at Owens College, Manchester, professor of political economy in University College, London, and an influential writer on finance and political economy, well known for his work on *The Coal Question* (1865).

[88] See Jevons, *Pure Logic or the Logic of Quality Apart from Quantity, with Remarks on Boole's System and on the Relation of Logic and Mathematics*, London, 1864, 87 p. For comments on Boole's system, see pages 69–87.

Jevons had outlined[89] a mode of thinking, a relationship[90] between logic and mathematics which a quarter century later Yule used as the basis for his researches on association and contingency.

This is the background against which the measures of association and contingency should be viewed. The preceding paragraphs are in no sense an attempt to outline the history of symbolic logic, but are designed to indicate the general nature of certain particular papers which led to the studies on association begun by Yule shortly before 1900. Treatises on symbolic logic written since 1900 cannot be considered to have had any profound influence on the course of statistical methods. Measures of association and contingency will be mentioned in the annotated bibliography beginning on page 130.

Correlation between Ranks. The first work on correlation by the "method of rank differences" is generally attributed to Spearman, who proposed the formula

$$R = 1 - \frac{3Sd}{n^2 - 1}$$

in 1904, in his article on "The Proof and Measurement of Association between Two Things." In a footnote on page 86, where he gives this formula, he has the following historical note: "This general idea seems to have been first due to Binet and Henri ("La fatigue intellectuelle," pp. 252–261), who, however, do not work it out far enough to obtain any definite measure of

[89] *Pure Logic*, p. 87.

The Substitution of Similars, the True Principles of Reasoning, London 1869. See page 86.

"On a General System of Numerically Definite Reasoning," *Memoirs of the Literary and Philosophical Society of Manchester*, Series 3, IV (1871), 330–352.

Principles of Science, A Treatise on Logic and Scientific Method, London, 1874, 1877, 1883, 1887, 1892, 1900, 1905.

[90] "Logical method must undoubtedly be the root of all scientific demonstration and of all sound thought in the common affairs of life; yet we find the most opposite and contradictory opinions held by different logicians as to the nature of the reasoning process. Metaphysical speculation will never remedy the present deplorable condition of the science. . . . . I hold that logic can only be regenerated by those who will render themselves acquainted with the exact methods of research which lead to undoubted truths in the mathematical and physical sciences. . . . . In this paper I have attempted to show that questions do exist in which logical and numerical methods coalesce and lend mutual aid." (From the paper "On a General System of Numerically Definite Reasoning," p. 352.)

correlation. Accordingly, Binet makes little further attempt in later research (*L'annêe psychologique*, Vol. IV) to render it of service, and soon appears to have altogether dropped it."

"The same idea occurred to myself and was developed as above, without being at the time acquainted with the previous work in this direction by Binet and Henri. In obtaining the above formula, I was greatly assisted by Dr. G. Lipps' showing generally that when an urn contains n balls numbered 1, 2, 3, ... n respectively; and when they are all drawn in turn (without being replaced); and when the difference is each time noted between the number on the ball and the order of its drawing; then the most probable (or middle) total sum of such differences added together without regard to sign, will be $\frac{n-1}{3}$. Previously I had only calculated this value for each particular size of n required by myself. Prof. Hausdorff further showed, generally, that such sum of differences will present a mean square deviation (from the above most probable value) $= \sqrt{\frac{(n+1)(2n+1)}{45}}$."

A comment made by Pearson in regard to Galton's first illustration of correlation (*q.v*) is also of interest in this connection. Galton had been making investigations into the inheritance of size in sweet-pea seeds, and had classified the seeds in five groups, ranging from the smallest to the largest. Pearson says that "There is little doubt that rank with rank was the first way in which he approached correlation. . . . . It is, I think, sufficient evidence that Galton dealt with the correlation of ranks before he even reached the correlation of variates, and the claim that it is a contribution of the psychologists some thirty or forty years later to the conception of correlation does not seem to me valid. Galton, we may with high probability suggest, had satisfied himself that the correlation of ranks was more cumbersome than the correlation of variates, because in the simplest case, that of the normal distribution, it fails to provide linear regression, but gives a non-integrable curve, which can only be plotted by aid of the probability integral table." (From the *Life of Francis Galton*, II, p. 393.)

Measures of Relationship Based upon the Pearson Product-Moment. From time to time, numerous coefficients have been proposed which are equivalent to the Pearson r under certain restricted conditions, or which are derived from it by the imposition of certain assumptions. For example, the correlation ratio, eta, is identical with r when the means of the arrays lie on straight lines, and rho is derivable from r when ranks may properly

be considered as deviates. These will not be treated individually here, but are mentioned in the annotated bibliography which follows.

Certain devices for computing the product-moment r sometimes appear to the unitiated as distinct methods of measuring correlation, whereas they are actually mere variant forms, algebraically equivalent to the Pearson r. Symonds has compiled a list of the different ways in which the Pearson r is commonly written.[91] A discussion of such computation methods does not properly belong in this section, but because one of these forms is frequently spoken of as though it were a method essentially different from the Pearson r, because it has often been rediscovered, and because there has been some question as to which of several recent writers is the originator of it, it seems worth while to sketch its early history. The formula is:

$$r_{xy} = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_v^2}{2\sigma_x\sigma_y}$$

where v is the difference between x and y. This was one of a general series of formulas presented by Pearson in 1896.[92] It was used by him in 1902 in work on the personal equation,[93] and in 1906 in a study of wasps.[94] It is referred to in the article "On Further Methods of Measuring Correlation,"[95] and Pearson's own comment[96] is that "It is quite reliable and often convenient."

The formula was published again by Boas in a letter to *Science*, on the

[91] "Variations of the Product-Moment (Pearson) Coefficient of Correlation," *Journal of Educational Psychology*, XVII (1926), 458–469.

[92] "We see, then, that the coefficient of correlation may be found from

$$r = \frac{\Sigma^2 - f_x^2\sigma_1^2 - f_y^2\sigma_2^2}{2f_xf_y\sigma_1\sigma_2}$$

or by calculating standard deviations." "Regression Heredity, and Panmixia," *Philosophical Transactions*, A, CLXXXVII, p. 279.

[93] See paper "On the Mathematical Theory of Errors of Judgment with Special Reference to the Personal Equation," *Philosophical Transactions*, A, CXCVIII (1902), 235–299. The formula appears on page 242, written

$$r_{23} = \frac{\sigma_{02}^2 + \sigma_{03}^2 - \sigma_{23}^2}{2\sigma_{02}\sigma_{03}},$$

where σ_{02} and σ_{03} are the standard deviations of the absolute judgments for individuals (2) and (3), and σ_{23} is the standard deviation of the relative judgments, that is, the difference in the judgments.

[94] *Biometrika*, V, p. 409.

[95] *Biometrika*, VI, p. 438.

[96] Letter to *Science*, XXX (1909), 23–34, in reply to a letter from Boas in the previous volume.

"Determination of the Coefficient of Correlation."[97] It is now much used as a computing device, and is the basis of the correlation charts devised by Toops, and by Otis, and of the check on the Kelley chart.

10. BIBLIOGRAPHY OF MEMOIRS RELATING PRIMARILY TO MEASURES OF CORRELATION OTHER THAN THE PEARSON PRODUCT-MOMENT

A full discussion of all the measures which have been proposed to measure the degree of relationship between two traits might well constitute a small volume. We are here presenting a chronological bibliography of the more important papers dealing with this subject, annotating where there is an item of special interest not suggested by the title of the paper. In cases where the title adequately indicates the type of material contained in the memoir, no comments have been added.

1832 QUETELET, A., "Sur la possibilité de mesurer l'influence des causes qui modifient les élémens sociaux," a letter to M. Willermé of the Institut de France, dated at Brussels, 1832. Yule says that this letter is now in the Library of the Royal Statistical Society.

"The method of serial chances seems to have been first brought forward, as a definite statistical method, by Quetelet, in a pamphlet published in 1832 . . . . but it is either explicitly or implicitly used in most statistical discussions of causation. Quetelet . . . . used as a measure of the 'degree of influence' of the cause, the function

$$\phi = \frac{(AB)(B) - (A)/(U)}{(A)/(U)}."$$

Yule, "On the Association of Attributes in Statistics," p. 281. Yule speaks of this formula as "Quetelet's function."

1899 SHEPPARD, W. F., "On the Application of the Theory of Error to Cases of Normal Distribution and Correlation," *Philosophical Transactions*, A, CXCII, 101–167.

Sheppard's test of independence of two distributions, page 128, is closely related to Yule's coefficient of association (1900) and to Pearson's χ^2 test of goodness of fit (1900). On page 141 is a formula later known as the "percentage of unlike-signed pairs."[98] This formula is quoted repeatedly by Pearson (1901) and by Yule (1900). Sheppard writes it:

$$\text{Divergence} = \frac{P}{P+R}.$$

1900 YULE, G. U., "On the Association of Attributes in Statistics: with Illustrations from the Material from the Childhood Society, &c," *Philosophical Transactions*, A, CXCIV, 257–319. Received October 20, and read December 7, 1899.

[97] *Science*, XXIX (1909), 823.

[98] Thorndike, *Mental and Social Measurements*, 1913 edition, p. 158.

"The classical writings on the subject are, I suppose, those of De Morgan Boole, and Jevons. Without attempting to criticise the work of his predecessors, to both of whom he was of course greatly indebted, the method of the latter must be allowed to far exceed theirs in clearness and simplicity. . . . . It is a matter of surprise to me that Jevons never made any practical application of his method (so far as I am aware) during the decade or more that elapsed between the publication of his paper . . . . and his death" (p. 258).

"Now it seems to me that one of the chief needs in handling statistics of the kind we are considering is some sort of a 'coefficient of association' which should take the place of the 'coefficient of correlation' for continuous variables, and be a measure of the approach of association towards complete independence on the one hand and complete association on the other." This Yule says must have the value zero when there is complete independence and ± 1 when there is complete association. Using Jevons' notation, but substituting Greek letters for his italics, Yule proposes

$$Q = \frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)},$$

and calls it the coefficient of association,[99] or Q. Yule also worked out the probable error of this Q. The letter Q was chosen as being the initial letter of "Quetelet."

Pearson, K., "On the Criterion that a given System of Deviations from the Probable in the Case of a Correlated System of Variables is such that it can be reasonably supposed to have arisen from Random Sampling," *Philosophical Magazine*, 5th series L, 157–175.

Here is presented the Chi-Square test of goodness of fit, later elaborated into the mean square contingency coefficient.

1901 Pearson, K., "On the Correlation of Characters not Quantitatively Measurable," *Philosophical Transactions*, A, CXCV, 1–47.

"In August, 1899, I presented a memoir to the Royal Society on the inheritance of coat-colour in the horse and of eye-colour in man, which was read November, 1899, and ultimately ordered to be published in the 'Phil. Trans.' Before that memoir was printed, Mr. Yule's valuable memoir on Association was read, and further, Mr. Leslie Bramley-Moore showed me that the theory of my memoir as given in § 6 of the present memoir led to somewhat divergent results according to the methods of proportioning adopted. We therefore undertook a new investigation of the theory of the whole subject, which is embodied in the present memoir. . . . . While I am responsible for the general outlines of the present paper, the rough draft of it was taken up and carried on in leisure moments by Mr. Leslie Bramley-Moore, Mr. L. N. G. Filon, M.A., and Miss Alice Lee, D. Sc." Page 1, *supra*.

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[99] In Pearson's notation, this would be $Q = \dfrac{ad - bc}{ad + bc}$. See page 132.

The formula later known as *tetrachoric r* is derived in this paper. Pearson assumes a surface of normal correlation divided into four parts by planes at right angles to the axes of the two distributions and at distances h and k from the origin, the units of measure being the standard deviations of the distributions. The frequency in the four divisions of the table he terms a, b, c, and d (a notation which for the sake of simplicity we will employ throughout the remainder of this discussion). He shows that if the frequencies a, b, c, and d are known, the distances h and k can be found, and that when the planes pass through the means, $r = \cos \pi \frac{b}{a+b}$, "which agrees with a result of Mr. Sheppard's, 'Phil. Trans.' A, vol. 192, p. 141. We have accordingly reached a generalised form of his result for *any* class-index whatever." Pearson derives the probable error of this r, and finds an expression for the correlation between errors in the position of the means of the two characters.

a	b
c	d

Pearson suggests four empirical expressions, called Q_1, Q_3, Q_4, and Q_5, which meet the conditions laid down by Yule as the essentials of a satisfactory coefficient of association, and adds that "Only by actual examination of the numerical results has it seemed possible to pick out the most efficient of the coefficients. Q_1 was found of little service." The numerical investigation which he made led him to conclude that ". . . . notwithstanding the extreme elegance and simplicity of Mr. Yule's coefficient of association Q_2, the coefficients Q_3, Q_4, and Q_5, which also satisfy his requirements, are much nearer to the values assumed by the correlation."

——— (with the assistance of Alice Lee, Ernest Warren, Agnes Fry, Cicily Fawcett, and others), "On the Principle of Homotyposis and its Relation to Heredity, to the Variability of the Individual, and to that of the Race. Part I. Homotyposis in the Vegetable Kingdom," *Philosophical Transactions*, A, CXCVII, 285–379. (Received October 6, read November 15, 1900.)

This gives an instance (page 293) of correlation between all possible pairs that can be formed from n individuals. The correlation, however, is computed by the product-moment method.

Yule, G. U., "On the Theory of Consistence of Logical Class-frequencies, and its Geometrical Representation," *Philosophical Transactions*, A, CXCVII, 91–134.

1902 Pearson, K., "On the Inheritance of Characters not Capable of Exact Quantitative Measurement," *Philosophical Transactions*, A, CXCV, 79–150.

This is really a continuation of the paper on the correlations of such characters, and illustrates the use of the new measures.

1903 Yule, G. U., "Notes on the Theory of Association of Attributes in Statistics," *Biometrika*, II, 121–134.

Much of the material in the early part of Yule's *Introduction to the Theory of Statistics* was first presented in this paper.

1904 PEARSON, K., "On a Generalised Theory of Alternative Inheritance, with Special Reference to Mendel's Laws," *Philosophical Transactions*, A, CCIII.

That Yule attached great importance to this memoir by Pearson is evident from a paragraph in the former's "Methods of Measuring Association," (1912), in which he says: "We next come to a whole series of researches on the mathematical theory of Mendelian inheritance . . . . in which the fourfold table occurs as a possible limiting case, either implied or expressed. The fundamental memoir is that by Professor Pearson above. . . . . It was this memoir by Professor Pearson which first directed my attention to the product-sum correlation for a fourfold table" (p. 607).

PEARSON, K., "On the Theory of Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series, I.*

"Hitherto, in order to obtain a measure of the degree of correlation or association, we have proceeded on the assumption that it was necessary to arrange the system of classes . . . . in some order, which corresponded to a real quantitative scale in the attribute, although we were unable to use this scale directly. . . . . The conception, however, of order in the classification, was at times very hampering. . . . . The object of this present paper is to deal with this novel conception of what I have termed *contingency*, and to see its relation to our older notions of association and normal correlation. The great value of the idea of contigency for economic, social, and biometric statistics seems to me to lie in the fact that it frees us from the need of determining scales before classifying our attributes." Referring to the measure developed in his own paper of 1900, Pearson says that "we can deduce a quantity P from χ^2 which is the probability that in any trial a system . . . . of observed frequencies will occur, which deviates more from [the system of theoretical frequencies] . . . . than the actually observed system does." "P would measure how far the observed system is or is not compatible with a basis of independent probability. . . . Hence $1 - P$ would be a proper measure of the contingency. I propose to call $1 - P$ the *contingency grade* . . . . I shall call $\phi^2 = \chi^2/N$ the *mean square contingency*." The symbol ψ, is used for the mean contingency obtained by summing for positive contingencies only. Then, for a normal distribution of frequency, the mean square contingency is $C_1 = \sqrt{\frac{\phi^2}{1 + \phi^2}}$ and "Provided our classes are sufficiently small to allow of us legitimately replacing by groupings over small areas the theoretical integrations, the coefficient of correlation can be found from the mean square contingency." The mean contingency coefficient, or C_2 is to be found from the mean contingency by the use of a table provided here.

SPEARMAN, C., "The Proof and Measurement of Association between Two Things," *American Journal of Psychology*, XV, 72–101.

Spearman proposes the "method of rank differences" (p. 86) using the formula

$$R = 1 - \frac{3\,Sd}{n - 1},$$

where Sd is the sum of the differences of rank for all the individuals, regardless of sign.

1905 BLAKEMAN, J., "On Tests for Linearity of Regression in Frequency Distributions," *Biometrika*, IV, 332–350.

LIPPS, G. F., "Die Bestimmung der Abhängigkeit zwischen den Merkmalen eines Gegenstandes," *Berichte der Mathematische-Physichen Klasse der königlich Sächsischen Gesellschaft der Wissenschaften.*

Yule says that Lipps here suggests a formula similar to the Q he himself proposed in 1900.

PEARSON, K., "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-linear Regression." *Drapers' Company Research Memoirs, Biometric Series*, II.

The idea of correlation is generalized and the correlation ratio developed. Several theorems concerning the correlations between various constants of the correlation surface are proved. The paper provides "a general method of dealing with the regression line and the variability of arrays in the case of skew correlation, without any assumptions as to the analytical form of the skew correlation surface." The probable error of the correlation ratio, eta, is determined.

1906 BLAKEMAN, J., AND PEARSON, K., "On the Probable Error of the Coefficient of Mean Square Contingency," *Biometrika*, V, 191–197.

JOHANNSEN, W., *Elemente der exacten Erblichkeitslehre*, Jena. (Reference from Yule and not verified.)

Yule: "In the same year, Johannsen, unaware of Pearson's use of the coefficient for a fourfold table gave a full and clear account of it in his 'Erblichkeitslehre,' an account which has largely influenced my own description."[100]

PEARSON, K., "On certain Points connected with Scale Order in the Case of the Correlation of two Characters which for some arrangement give a Linear Regression Line," *Biometrika*, V, 176–178.

[100] "On the Methods of Measuring Association," p. 609.

SPEARMAN, C., "'Footrule' for Measuring Correlation," *British Journal of Psychology*, II, 89–108.

Spearman speaks of the need of a 'footrule', for computing correlation and of the utility for psychological investigations of comparison by ranks, and proposes again the formula he suggested in 1904. He says that if the usual product moment method is applied to ranks we get $r = 1 - \frac{\Sigma d^2}{m}$, where d is the numerical difference between any corresponding pair of ranks, and m is the mean value of d by mere chance, being $= \frac{n(n-1)}{6}$. He recommends the shorter form, saying that in psychology "the squaring is likely to do more harm than good."

YULE, G. U., "On a Property which holds good for all Groupings of a Normal Distribution of Frequency for Two Variables, with Applications to the study of Contingency Tables for the Inheritance of Unmeasured Qualities," *Proceedings of the Royal Society*, A, LXXVII.

——— "On the Influence of Bias and of Personal Equation in Statistics of Ill-defined Qualities," *Journal of the Anthropological Institute*, XXXVI, 325–381.

An abstract is to be found in the *Proceedings of the Royal Society*, A, LXXVII, 337.

1907 PEARSON, K., "Reply to certain Criticisms of Mr. G. U. Yule," *Biometrika*, V (1906–07), 470–476.

This is a further discussion of association and contingency, particularly addressed to Yule's two papers in the *Proceedings of the Royal Society* for 1906.

——— "On Further Methods of Determining Correlation," *Drapers' Company Research Memoirs, Biometric Series*, IV.

Pearson gives here the variate difference method[101] with comment that "This method of finding r_{xy} has long been in use as an alternative method to the product-moment method," and derives a special formula, assuming a normal distribution, where the two quantities have the same mean and the same standard deviation, in which case "the coefficient of correlation is the result of subtracting from unity π times the square of the mean sum of the positive differences of paired variates divided by their common standard deviation." He discusses the general nature of correlation among grades (or percentiles) and among ranks, says that a grade is an index to the variate but not an independent character of the variate, and that Spearman has correlated ranks as though they were true values of the variate. Pearson thinks it a retrograde step to make rank a unit in itself. He proves that rho, $\rho = 1 - \frac{6\,\Sigma d^2}{n(n^2-1)}$, is the product moment for grades, and that $r = 2 \sin \frac{\pi}{6} \rho$. He gives two rank correlation formulas, one for ranks and one for true grades,

[101] See pages 129 and 189.

and says that ". . . . variate correlation found by ranks may prove to be a useful method of dealing with correlation when it is needful to give a rough answer to a problem in a brief time, or when the material itself is incapable of being accurately measured. In all such cases, mean square or rank differences will be more accurate than mean positive differences. But both methods must be used with caution, and their easy application must not lead us to approve exaggerated statements as to their accuracy." Pearson also gives formulas for the difference method of finding the correlation of grades and of ranks.

SPEARMAN, C., "Demonstration of Formulae for True Measurement of Correlation," *American Journal of Psychology*, XVIII, 160–169.

THORNDIKE, E. L., *Empirical Studies in the Theory of Measurement.*

". . . . It is probable that unless these methods are soon subjected to a review by some one who can both make perfectly clear their presuppositions to the rank and file of investigators in psychology and the social sciences and prove their applicability to actual cases of relations to be measured, there will be damage done in two ways. Many investigators will as in the past use hopelessly crude methods and misinterpret relationships; and also many investigators will learn of the formulas of the mathematical statisticians and apply them to cases where they are out of place and give inadequate and misleading results. To both of these errors the writer, for instance, confesses himself guilty in the past. I am unable to make such a review but as no one of those who are able seems willing, I have made a partial and inferior substitute for it which I hope may, in so far as it is sound, be instructive to students of mental measurements and, in so far as it is unsound, may provoke some capable student to give the adequate review that is so much needed."

Thorndike advocates the use of the *Median-Ratio* instead of the Pearson r on the ground that the meaning of the former is unambiguous, that it suggests a relation with a direct meaning for each pair of values in the table rather than a meaning for the table as a whole, and that he thinks it better to use the slope of a line so placed that the sum of the positive and negative deviations from it shall be zero rather than that the sum of the squares of the deviations shall be a minimum.[102]

1909 BOAS, F., "Determination of the Coefficient of Correlation," a letter to *Science*, XXIX, N.S., 823.

[102] Note the comment by Kelley on page 545 of the number of the *Teachers College Record* devoted to the work of Thorndike (1926):

"It seems to the writer that the reason he developed these procedures lies in the fact that he has never made systematic use in the choice of statistics of the principle of least squares. Upon more than one occasion he has just fallen short of making explicit and utilizing the principle of least absolute error or that of least median error. That without one of these principles as guide in derivation his techniques have been so essentially sound is due to his keen appreciation of probable and systematic errors in the resulting statistics. In short, he has used the principle of minimal error as an agency in proof but not in derivation."

Harris, J. A., "The Correlation between a Variable and the Deviation of a Dependent Variable from its Probable Value," *Biometrika*, VI (1908–09), 438–443.

Pearson, K., Letter on "The Determination of the Coefficient of Correlation," a reply to Boas, *Science*, XXX, 23.

——— "On a New Method of Determining Correlation between a measured Character *A* and a Character *B*, of which only the Percentage of cases wherein *B* exceeds (or falls short of) a given Intensity is recorded for each grade of *A*," *Biometrika*, VII, 96–105.

The method of bi-serial *r* is here described but not named.

1910 Brownlee, J., "The Significance of the Correlation Coefficient when applied to Mendelian Distributions," *Proceedings of the Royal Society of Edinburgh*, XXX, 473–507.

Everitt, F. P., "Tables of the Tetrachoric Functions for Fourfold Correlation Tables," *Biometrika*, VII, 437–451.

Pearson, K., "On a New Method of Determining Correlation, when one Variable is given by Alternative and the other by Multiple Categories," *Biometrika*, VII, 248–257.

This paper presents the method of bi-serial eta, but does not use that term.

1911 Heron, D., "The Danger of Certain Formulae suggested as Substitutes for the Correlation Coefficient," *Biometrika*, VIII (1911–12), 109–122.

This is mainly a criticism of Yule's Coefficient of Association.

Pearson, K., "On a Correction to be Made to the Correlation Ratio η," *Biometrika*, VIII (1911–12), 254–256.

Yule, G. U., *Introduction to the Theory of Statistics*, 1st edition.

This has a marked emphasis upon methods of association and contingency, a discussion of the correlation ratio, and a page devoted to the correlation for a fourfold table.

1912 Everitt, P. F., "Supplementary Tables for Finding the Correlation Coefficient from Tetrachoric Groupings," *Biometricka*, VIII (1911–12), 385–395.

Yule, G. U., "On the Methods of Measuring Association between Two Attributes," *Journal of the Royal Statistical Society*, LXXV, 579–642.

This memoir presents a survey of previous papers on the subject of association, and of investigations employing the coefficient of association, the coefficient of correlation for a four-fold table, or Pearson's tetrachoric *r*, called

here the normal coefficient. It gives a bibliography of thirty-six titles to which reference has been made in the body of the paper. Yule proposes here his *coefficient of colligation.*

1913 Harris, J. A., "On the Calculation of Intra-Class and Inter-Class Coefficients of Correlation from Class Moments when the Number of Possible Combinations is large," *Biometrika*, IX, 446–472.

Pearson, K., "On the Probable Error of a Coefficient of Correlation as found from a Fourfold Table," *Biometrika*, IX, 22–27.

——— "On the Measurement of the Influence of 'Broad Categories' on Correlation," *Biometrika*, IX, 116–139.

Pearson, K., and Heron, D., "On Theories of Association," *Biometrika*, IX, 159–315.

This is a reply to Yule's paper of 1912, and the writers "believe that, if Mr. Yule's views are accepted, irreparable damage will be done to the growth of modern statistical theory," and say ". . . . we shall term Mr. Yule's latest method of approaching the problem of relationship of attributes the *method of pseudo-ranks.* In addition we deal incidentally with other coefficients and reply to certain criticisms, not to say charges, Mr. Yule has made against the work of one or both of us." The first nine pages are devoted to a critical analysis of previous papers by Yule, Pearson, Boas, and others, thus furnishing a history of the subject quite different from that given by Yule the previous year.

Pearson, K., "On the Surface of Constant Association $Q = 0.6$." *Biometrika*, IX, 534–537.

"Student," "The Correction to be made to the Correlation Ratio for Grouping," *Biometrika*, IX, 316–320.

1914 Cave, Beatrice, M. and Pearson, K., "Numerical Illustrations of the Variate Difference Correlation Method," *Biometrika*, X (1914–15), 340–355.

Harris, J. A., "On Spurious Values of Intra-class Correlation Coefficients arising from Disorderly Differentiation within the Classes," *Biometrika*, X (1914–15), 412–416.

Isserlis, L., "On the Partial Correlation Ratio. Part I. Theoretical," *Biometrika*, X (1914–15), 391–411.

Pearson, K., "On an Extension of the Method of Correlation by Grades or Ranks," *Biometrika*, X (1914–15), 416–418.

Pearson's work of 1907 is here extended to cover the two cases where: (1) one variate is given quantitatively and the other by ranks and (2) one variate is given by broad categories and the other by ranks.

——— *Tables for Statisticians and Biometricians.*

SOPER, H. E., "On the Probable Error of the Bi-Serial Expression for the Correlation Coefficient," *Biometrika*, X (1914–15), 384–390.

1915 ISSERLIS, L., "On the Partial Correlation Ratio, Part II, Numerical," *Biometrika*, XI, 50–66.

PEARSON, K., "On the Probable Error of a Contingency Coefficient," *Biometrika* X, 570–574.

RITCHIE-SCOTT, A., "Note on the Probable Error of the Coefficient of Correlation in the Variate Difference Correlation Method," *Biometrika*, XI, 136–138.

1916 OTIS, A. S., "The Reliability of Spelling Scales," *School and Society*, IV.

Otis defines the *curve of relation* as "that line (presumably of but one curvature) in which it is assumed the points would all lie if there were perfect correlation between the two sets of values. It may be thought of, roughly, as the curve which is nearest to all the points as plotted." He proposes the *curve of correspondence by rank* or *curve of rank relation*, equivalent to r for a normal distribution, with linear regression. Otis says this formula is similar to Spearman's *Footrule* and to "Yule's eta," but considers it superior to either.

ISSERLIS, L., "On certain Probable Errors and Correlation Coefficients of Multiple Frequency Distributions with Skew Regression," *Biometrika*, XI, 185–190.

PEARSON, K., "On the General Theory of Multiple Contingency with Special Reference to Partial Contingency," *Biometrika*, XI, 145–158.

——— "On some Novel Properties of Partial and Multiple Correlation Coefficients in a Universe of Manifold Characteristics," *Biometrika*, XI, 231–238.

YOUNG, A., AND PEARSON, K., "On the Probable Error of a Coefficient of Contingency without Approximation, *Biometrika*, XI, 215–230.

1917 LEE, ALICE, "Further Supplementary Tables for Determining High Correlations from Tetrachoric Groupings," *Biometrika*, XI, 284–291.

PEARSON, K., "On the Probable Error of Biserial η," *Biometrika*, XI, 292–302.

PERSONS, W. M., "On the Variate Difference-Correlation Method and Curve Fitting," *Quarterly Publications of the American Statistical Association*, XVI.

1918 Isserlis, L., "On a Formula for the Product-Moment Coefficient of any Order of a Normal Frequency Distribution in any Number of Variables," *Biometrika*, XII, 134–139.

Ritchie-Scott, A., "The Correlation Coefficient of a Polychoric Table," *Biometrika*, XII (1918–19), 93–133.

1920 Pearson, K., "Second Note on the Coefficient of Correlation as determined from the Quantitative Measurement of One Variate and the Ranking of a Second Variate," *Biometrika*, XIII, 302–305.

"Student," "An Experimental Determination of the Probable Error of Dr. Spearman's correlation coefficients," *Biometrika*, XIII, 263–282.

1922 Henderson, J., "On Expansions in Tetrachoric Functions," *Biometrika*, XIV (1922–23), 157–192.

Otis, A. S., "The Method for Finding the Correspondence between Scores in Two Tests," *Journal of Educational Psychology*, XIII, 529–544.

Otis states that "the equation of the line which most probably expresses the true relationship between x and y is

$$y = \frac{\sigma_y}{\sigma_x}\ x,\text{"}$$

a statement which he says has been challenged by eminent statisticians. He offers first a proof by analogy, and then an analytical proof on the assumption that $r_{xy} = 1$. He says that if r_{xx} and r_{yy} are the two reliability coefficients, then the most probable relation is

$$y = \sqrt{\frac{r_{yy}}{r_{xx}}} \cdot \frac{\sigma_y}{\sigma_x}\, x.$$

Pearson, K., and Pearson, E., "On Polychoric Coefficients of Correlation," *Biometrika*, XIV, 127–156.

1923 Anderson, O., "Über ein neues Verfahren bei Anwendung der 'Variate-Difference' Methode," *Biometrika*, XV, 134–172.

Narumi, S., "On the General Forms of Bivariate Frequency Distributions which are Mathematically Possible when Regression and Variation are subjected to Limiting Conditions," *Biometrika*, XV, 77–88, 209–221.

Pearson, E., "The Probable Error of a class-Index Correlation," *Biometrika* XIV, 261–280.

Pearson, K., "Notes on Skew Frequency Surfaces," *Biometrika*, XV, 222–230.

——— "On Non-Skew Frequency Surfaces," *Biometrika*, XV.

——— "On the Correction necessary for the Correlation Ratio η," *Biometrika*, XIV (1922–23), 412–417.

——— "On the Relationship of Health to the Psychical and Physical Characters in School Children," *Drapers' Company Research Memoirs, Studies in National Deterioration*, IV.

Biserial eta, triserial eta, biserial r, and tetrachoric r are used in the analysis of data on the health of children. Unusual graphic methods are also used.

PEARSON, K., AND ELDERTON, ETHEL, "On the Variate Difference Method," *Biometrika*, XIV (1922–23), 281–310.

RHODES, E. C., "On a certain Skew Correlation Surface," *Biometrika*, XIV (1922–23), 355–377.

1924 EZEKIEL, M. J. B., "A Method of Handling Curvilinear Correlation for any Number of Variables," *Journal of the American Statistical Association*, XIX, 431–453.

JACKSON, D., "The Algebra of Correlation," *American Mathematical Monthly*, XXXI, 110–121.

RIDER, P. R., "The Correlation between two Variates one of which is Normally Distributed," *American Mathematical Monthly*, XXXI.

1925 LEE, ALICE, "Tables of the First Twenty Tetrachoric Functions to Seven Decimal Places," *Biometrika*, XVII, 343–354.

NEWBOLD, ETHEL M., "Notes on an Experimental Test of Errors in Partial Correlation Coefficients, derived from Fourfold and Biserial Total Coefficients," *Biometrika*, XVII, 251–265.

PEARSON, K., "Note on Miss Newbold's Memoir," *Biometrika*, XVII, 266–267.

RHODES, E. C., "On a Skew Correlation Surface," *Biometrika*, XVII

YASUKAWA, K., "On the Means, Standard Deviations, Correlations and Frequency Distributions of Functions of Variates, *Biometrika*, XVII, 211–237.

1926 MUSSELMAN, J. R., "On the linear Correlation Ratio in the Case of Certain Symmetrical Frequency Distributions," *Biometrika*, XVIII, 228–230.

SPLAWA-NEYMAN, J., "Further Notes on Non-linear Regression," *Biometrika*, XVIII, 257–262.

1927 McCLOY, C. H., "A Method of Computing Partial Correlation and Regression Equations with Variables having curvilinear Intercorrelations," *Journal of Educational Research* XVI, 285–295.

CHAPTER VI

THE THEORY OF TWO FACTORS

Purpose of the Chapter. Spearman's Theory of Two Factors[1] is one of the most striking illustrations of an educational and psychological hypothesis which has been defended and attacked almost solely by statistical arguments. The importance of the hypothesis to educational practice, the verifying or discrediting of it by statistical procedure, and the criteria for that statistical procedure, are all closely interrelated. Since a full treatment of the matter is not consistent with the general purpose and scope of the present study, and since any treatment which discusses the statistical aspects and ignores the psychological implications is necessarily inadequate, it has seemed best to present merely an annotated bibliography, chronologically arranged.

BIBLIOGRAPHY

1904 SPEARMAN, C., "General Intelligence Objectively Determined and Measured," *American Journal of Psychology*, XV (1904), 201–293.

Spearman enunciates his theory of a general factor in intellectual abilities, and offers a first criterion for the divisibility of measurements into two factors. He also presents here the idea of a "Hierarchy of Intelligences."

1906 SPEARMAN, C. AND KRUEGER, F., "Die Korrelation zwischen verschiedenen geistigen Leistungsfähigkeiten," *Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, XLIV, Part I, Leipzig, 1907, 50–114. (Paper received September 8, 1906.)

A second criterion is proposed, namely

$$\sqrt{r_{p_1p_2}r_{q_1q_2}} = \sqrt[4]{r_{p_1q_1}r_{p_1q_2}r_{p_2q_1}r_{p_2q_2}}.$$

A correlation coefficient is assumed to be zero unless it is more than twice its probable error. Coefficients of correlation are based on eleven cases. A hierarchy is said to be a necessary consequence of the existence of a single general factor in mental traits.

1909 THORNDIKE, E. L., LAY, W. AND DEAN, P. R., "The Relation of Accuracy in Sensory Discrimination to General Intelligence, "*American Journal of Psychology*, XX (1909), 364–369.

[1] The name *Theory of Two Factors* was suggested by Sancti de Sanctis in 1913. See, *Proceedings of the International Congress of Medicine, Section for Psychiatry* (London, 1913).

Thorndike concludes that: "The measurements obtained in the present investigation do not in the least support this hypothesis of approximately perfect correspondence between 'General Discrimination' and 'General Intelligence.' . . . . It is perhaps best to wait for further and fuller measurements of the relation in question before attempting to explain the difference between this result and Spearman's. . . . . I may add that other studies of correlation made by my students and myself are unanimous in contradicting Spearman's ingenious hypothesis of one sole common element as the cause of all positive correlation."

1910 BURT, C., "Experimental Tests of General Intelligence," *British Journal of Psychology*, III (1910), 94–177.

This is a study of experiments carried on in 1907–08 in which twelve different tests were given to forty-three twelve-year old boys, and in general the author's conclusions agree with those of Spearman. He publishes here a third criterion which Spearman had communicated to Burt in a letter.

BROWN, W., "Some Experimental Results in the Correlation of Mental Abilities," *British Journal of Psychology*, III (1910), 296–322.

Brown had computed correlations and had constructed tables of the coefficients, arranging the tests in the order of magnitude of the average correlation of each with all the rest. His conclusion is that, "The first thing to be noted in the groups of coefficients arranged in this way is that *not one of them show the 'hierarchical arrangement'* [Italics as in original]. . . . . A definite solution of the question of the existence or non-existence of one central mental ability is yet to be sought."

1911 ABELSON, A. R., "The Measurement of the Mental Ability of 'Backward' Children," *British Journal of Psychology*, IV (1911), 268–314.

This is a report of research carried out under the direction of Spearman.

1912 SIMPSON, B. R., *Correlations of Mental Abilities*, Teachers College, Columbia University, Contributions to Education.

This study deals with the mental abilities of seventeen college professors and of twenty men from New York who had never held any positions demanding a high grade of intelligence. The results were inconclusive, but Simpson was unable to detect any tendency toward hierarchical order.

1913 SPEARMAN, C. AND HART, B., "General Ability, Its Existence and Nature," *British Journal of Psychology*, V (1913), 51–84. The paper was received November 1, 1911.

This was probably the first extensive discussion of the physiological and psychological considerations attendant upon the theory of general intellective energy. A mathematical appendix to the paper offered a proof that when the tetrad equation is satisfied, the correlation between any two columns will be unity, and gave a formula to correct the intercolumnar correlation for errors of sampling in the r's on which it is based.

WYATT, S., "The Quantitative Investigation of Higher Mental Processes," *British Journal of Psychology*, VI (1913), 109–133.

The author concludes that " . . . . the correlation between the pairs of columns is high and positive, and the contention of Hart and Spearman that a General Common Factor exists receives further support."

1914 SPEARMAN, C., "The Theory of Two Factors," *Psychological Review*, XXI (101–115).

Spearman proposes another criterion, deduced from the partial coefficient of correlation between trait A and trait B with the general factor held constant. He also reviews Simpson's study and finds in it evidence of a general factor.

1915 WEBB, E., "Character and Intelligence," *British Journal of Psychology, Monograph Supplement*, I (1915), Number 3.

This was a doctor's thesis written in University College under Spearman's guidance, and some of the proofs were furnished by Spearman. Webb says: "This result (an average intercolumnar correlation of 1.02 ± .08) is an additional item of evidence in support of the Theory of a General Factor of Intellective Energy. It takes its place in the huge array of evidence collected by Professor Spearman from experimental tests by many investigators—the steadiness of results being such as to rival the niceties which physical measurements reveal. It should be remembered that the raw material of our own (comparatively small) contribution to this total result consisted of test-papers numbering nearly ten-thousand." The paper presents the hypothesis that "a second factor, of wide generality, exists; and that this factor is prominent on the 'character' side of mental activity," suggests that this factor might be "persistence of motives" or "consistency of action resulting from deliberate volition, or will" and uses the symbol 'w' to indicate this factor.

1916 McCALL, W. A., *Correlation of Some Psychological and Educational Measurements*, Teachers College, Columbia University, Contributions to Education.

Although he expresses his conclusions with some reservations, in general McCall says that his results cast much doubt upon the existence of a general factor.

SPEARMAN, C., "Some Comments on Mr. Thomson's Paper," *British Journal of Psychology*, VIII (1915–17), 282–284.

THOMSON, G., "A Hierarchy without a General Factor," *British Journal of Psychology*, VIII (1915–17), 271–281.

The paper was prepared in 1914, but its publication was delayed at the author's request because both Spearman and Thomson were engaged in military work, and Spearman could not attend the meeting at which the paper was to have been presented. Thomson had succeeded in artificially producing a hierarchy from the results of a set of imitation mental tests which were known *a priori* to have no common factor, the "tests" being throws of dice. He concludes: "*There is therefore nothing to show whether the many cases brought*

forward by him [Spearman] *really contain a General Factor or not.* [Italics as in original.] It must not be hastily and illogically concluded by anyone that therefore General Ability is a fiction. Its existence or non-existence is, so far as the mathematical argument goes, an entirely open question. . . . . All I have shown is that Professor Spearman's calculations are incapable of discriminating between a General Factor and overlapping Group Factors."

1919 THOMSON, G., "The Proof or Disproof of the Existence of General Ability," *British Journal of Psychology*, IX (1919), 321–336. (Paper received December 10, 1918.)

——— "The Hierarchy of Abilities," *British Journal of Psychology*, IX (1919), 337–344. (Paper received December 17, 1918.)

——— "On the Degree of Perfection of Hierarchical Order among Correlation Coefficients," *Biometrika*, XII (1919), 355–366.

——— "On the Cause of Hierarchical Order among the Correlation Coefficients of a Number of Variates Taken in Pairs," *Proceedings of the Royal Society*, XCV, A (1919).

Thomson presents schematic arrangements of overlapping dice throws as an argument that group factors might cause correlations which might be erroneously ascribed to the presence of a central factor, and submits that if reasoning be fallacious in this simple case, it must *a fortiori* be unsound when applied to more complex data.

GARNETT, J. C. M., "General Ability, Cleverness and Purpose," "*British Journal of Psychology*, IX (1919), 345–366. (Paper received March 9, 1919.)

——— "On Certain Independent Factors in Mental Measurements," *Proceedings of the Royal Society*, XCVI (1919), 91–111.

In these two papers and in another of the following year (*infra*) Garnett sets forth the mathematical argument for the existence of a general factor. He postulates a trait "c" independent of "g" and distributed along an axis perpendicular to the axis of "g." He also carries the argument over into three dimensional space to account for a second group factor called "Purpose," similar to Webb's "Cleverness," and suggests that of the three, "g" may be the only one which is educable.

THOMSON, G. AND GARNETT, J. C. M., "Joint Note on 'The Hierarchy of Abilities,' " *British Journal of Psychology*, IX (1919), 367–368.

The correspondence carried on between Thomson and Garnett concerning the papers listed above, resulted in their making a statement about points on which they felt themselves in agreement.

1920 GARNETT, J. C. M., "The Single General Factor in Dissimilar Mental Measurements," *British Journal of Psychology*, X (1920), 242–258.

SPEARMAN, C., "Manifold Sub-Theories of 'The Two Factors,' " *Psychological Review*, XXVII (1920), 159–172.

This contains a summary of the arguments for a general factor, and a long reply to Thomson's "Hierarchy without a General Factor." The sub-theories which he mentions are Otis's view that each test has its own particular "spread" of elements, and Thomson's suggestion of two levels of mental processes.

THOMSON, G., "The General Factor Fallacy in Psychology," *British Journal of Psychology*, X (1919–20), 319–326.

——— "General versus Group Factors in Mental Activities," *Psychological Review*, XXVII (1920), 173–190.

This contains a summary of "various scattered papers which in the writer's opinion [Thomson's] prove the invalidity of the reasoning upon which Professor Spearman has based his Theory of General Ability. . . . ."

1921 GARNETT, J. C. M., *Education and World Citizenship, an Essay towards a Science of Education*, Cambridge.

1922 SPEARMAN, C., "Recent Contributions to the 'Theory of Two Factors,'" *British Journal of Psychology*, XIII (1922), 26–30.

1923 KELLEY, T. L., *Statistical Method*, pages 208–212, section on the probable error of a coefficient corrected for attenuation.

SPEARMAN, C., "Further Note on the 'Theory of Two Factors,'" *British Journal of Psychology*, XIII, 266–270.

THOMSON, G. H., "On 'Hierarchical Order' among Correlation Coefficients," *Biometrika*, XV, 150–160.

——— "Note on 'Hierarchical Order' among Correlation Coefficients," *British Journal of Psychology*, XIV, 218.

1924 SPEARMAN, C., AND THORNDIKE, E. L., "A Communication and a Reply," *Journal of Educational Psychology*, XV, 393–394.

SPEARMAN, C., AND HOLZINGER, K., "The Sampling Error in the 'Theory of Two Factors,'" *British Journal of Psychology*, XV, 17–19.

1925 SPEARMAN, C., AND HOLZINGER, K., "Note on the Sampling Error of Tetrad Differences," *British Journal of Psychology*, XVI, 86–88.

1926 SLOCOMBE, C. S., "The Constancy of 'g,' General Intelligence," *British Journal of Psychology*, XVII, 93–110.

1927 PEARSON, K., AND MOUL, MARGARET, "The Mathematics of Intelligence, I. The Sampling Errors in the Theory of a Generalized Factor," *Biometrika*, XIX, 246–291.

SPEARMAN, C., *The Nature of Intelligence and the Principles of Cognition.*

This book elaborates the psychological theory but does not deal with the statistical argument.

——— "Material *versus* Abstract Factors in Correlation," *British Journal of Psychology*, XVII, 322–326.

THOMSON, G. H., "The Tetrad-Difference Criterion," *British Journal of Psychology*, XVII, 235–255.

1928 DODD, S. C., "The Coefficient of Equiproportion as a Criterion of Hierarchy," *Journal of Educational Psychology*, XIX, 217–229.

KELLEY, T. L., *Crossroads in the Mind of Man, a Study of Differentiable Mental Abilities*, Leland Stanford University.

SPEARMAN, C., *The Abilities of Man.*

Here Spearman considers various statistical aspects of the Theory of Two Factors, and presents a good deal of new material relative to such problems as the measurement of a person's "g," measurement of a person's specific abilities and weighting to make the best team of tests.

CHAPTER VII

Statistics as a Subject of Instruction in American Universities

1. purpose and method of the chapter

Purpose. One of the striking developments in college curricula in recent years has been the rapid rise of statistical theory and method as an important subject of instruction. Among the indications of this are the remarkable increase in the number of colleges teaching it, the number of departments concerned, the number of courses offered within a single institution, and the number of students enrolled in these courses. Thirty years ago there were probably not more than a dozen universities in America in which a student could obtain instruction in statistical methods. At the present time almost every university in the country and most of the larger colleges have at least one or two courses in statistics, while some of the larger universities offer from ten to twenty-five courses in five or six different departments. It is the purpose of this chapter to present a picture of this growth and to attempt to trace the influences which have brought it about. In so doing the greater emphasis has been placed upon pioneer work. Courses which have been given in the last ten years, and in particular the work of men who are now engaged in teaching statistics, has been touched upon but briefly, or not at all.

Sources of Information. The statements made in the following paragraphs are based on four sources of information:

1. College catalogues.
2. A few magazine articles to which reference will be made later.
3. Personal conversations and correspondence.
4. The replies returned to a questionnaire sent out by the American Statistical Association in 1925.[1]

[1] At the time this study was made, these questionnaire blanks were on file in the office of Edwin W. Kopf, Assistant Statistician of the Metropolitan Life Insurance Company, where the writer examined them. For another study based on the same returns, see Glover, "Statistical Teaching in American Colleges and Universities," *Journal of the American Statistical Association*, XXI (1926), 419–424.

Method of Studying Catalogues. The method of studying the college catalogues requires some explanation. Certain institutions were selected for detailed study of the courses offered prior to 1920. These schools cannot be considered as typical of the country at large, but were selected because in each of them strong courses in statistics were known to have been offered at some time. A three-way distribution was made, in which the courses were arranged according to the year in which they were given, the institution, and the department offering them. It seemed satisfactory to classify all courses in one of the six general categories:

Anthropology (sometimes grouped by the catalogue with psychology, with sociology, or with education).

Biology (including zoölogy, botany, eugenics, thremmatology).

Economics (including sociology).

Education (often grouped with psychology).

Mathematics (including astronomy).

Psychology.

Sources of Unreliability in the Study. The data thus obtained could at the best indicate little more than trends and suggest to the writer the generalizations offered in the following paragraphs. To reproduce these statistical charts in this discussion would be to make an unwarranted claim of accuracy. For several reasons such a study could not be made entirely reliable without an amount of labor out of all proportion to the importance of the inquiry.[2] An examination of all the information thus collected seems to warrant the generalizations and observations that are set down later in the chapter.

2. INSTRUCTION IN STATISTICS IN 1890

Few College Courses in the Subject. The situation relative to the teaching of statistics in American Colleges in 1890 has been well described by Francis A. Walker[3] in a paper in which he calls in true prophetic fashion for the very development which has since taken place.[4]

[2] A brief statement of these reasons will be found in the appendix.

[3] Francis Amasa Walker (1840–1897) was the superintendent of the 10th census, president of the Massachusetts Institute of Technology, after 1881, and president of the American Statistical Association (1882–96). For a biography see Wright, "Francis Amasa Walker," *Publications of the American Statistical Association*, (1897), 245–275.

[4] "The Study of Statistics in Colleges and Technical Schools," *Technology Quarterly*, Feb. 1890, reprinted in a collection of his writings called *Discussions in Education*, 1899.

"During the past twenty or even ten years there has been an astonishingly rapid development of historical and economic studies in our higher institutions of learning. . . . . Unfortunately, while this rapid development of historical and economic work has been going on, a branch of study which has the highest virtue at once to train the hand of the historical or the economic scholar and to furnish him with professional tools of the first importance has been almost wholly neglected. I refer to statistics, whose very methods are hardly known to the great majority of our economists and historians; and which is still to have its first chair founded in an American college. There are, indeed, a few schools where a little elementary instruction has, of recent years, been given in the use of figures as a means of testing sociological conclusions; but in no one of them has a full proper course of statistics been established. It cannot be long, however, before the growing interest in economics and history will compel the recognition of statistics as a distinct and an important part of the curriculum of every progressive institution. The main difficulty will be to find the men who have had the training at once severe and liberal, which will qualify them to inspire and direct these studies. . . . .

"Instruction directly intended to qualify a student to use statistics, and to compile tables with ease, confidence, and accuracy, is now given at Harvard University, Columbia College, the Institute of Technology and probably elsewhere. The pupil is taught to look up the data relating to a given subject, as these may be found scattered through long series of official reports; to bring the various statements together, to examine them as to their proper comparability, to test their accuracy by all means which may be available, and to put them together into tables. The student is further taught to work out the percentages involved and to set one class of facts into relation with others; . . . . and finally to make diagrams or charts. . . . . In none of the higher institutions, however, is this branch of study carried as far as it ought to be, nor are all the methods of instruction in this department yet worked out to their greatest efficiency. Still, the good work has been well begun; and the constantly growing appreciation of the ability to compile and to use statistics for the purposes of political, economic, and social discussion cannot fail to cause a rapid development of this feature of the college course. . . . .

"The scope of this paper does not include a discussion of the subjects and the order of studies designed to give the investigator the power to discover statistically the laws which govern the action of social and economic forces. Such a course would necessarily be long and severe. For the best results it should embrace the highest mathematics of our American colleges, and

should be largely directed to the development of the biological sense. . . . . Far smaller still will be the numbers of those whose natural endowments and whose chosen pursuits would justify the long and laborious training, the patient practice, and the acquisition of the large and various learning, which alone can qualify the student of history, of sociology, or of political economy confidently and surely to educe from thousands of pages closely packed with figures some hitherto unsuspected law of human life or conduct."

Walker was untiring in his agitation for the organization of college courses in statistics. In his last public address,[5] made only five days before his death, his principal theme was the need of technical training for statisticians and the delinquency of the American college in failing to furnish it. "The United States have spent millions and tens of millions upon the collection, compilation, and publication of statistics, and yet they have never spent, perhaps $10,000—certainly the government has never spent anything—in training and preparing the men who should conduct the statistical service of the country. . . . . I do not believe that it is at all an exaggeration of the fact to say that if one per cent of what the U. S. Government has spent upon statistics had been devoted to the training and preparation of men to conduct our service, it would have saved at least one-third of the cost of all the statistics collected in the past, and would have enhanced and improved the quality of the results almost indefinitely. . . . . I do not know of a single man now holding, or who ever held, a position in this country as . . . . a statistician, who had any elementary training for his work."

3. FIRST COLLEGE COURSES IN STATISTICS

Course in Columbia College. The first course in statistics given in any American University was probably[6] taught by Professor Richmond Mayo-Smith at Columbia College in 1880, which was the year in which the School of Political Science was founded there. In 1884 the description of the course contained these words: "Finally is considered the method of statisti-

[5] "Remarks of President Walker at Washington," *Publications of the American Statistical Association*, V, (1896–97), 179–187.

[6] For a stretch of years about this time Columbia was issuing, not a formal catalogue, but a handbook in which was printed a synopsis of the courses of instruction. Among the courses for third-year students in 1880 is listed "Social Science and Statistics," the name of the instructor not being given. Mayo-Smith was on the faculty at that time as an Adjunct-Professor of Social Science. The course persisted with substantially the same description for many years, and when the formal printing of catalogues was resumed in 1894, Mayo-Smith's name is found appended to the description of the course.

cal observations, the value of the results obtained, the doctrine of free will, and the possibility of discovering social laws."

Courses in the University of Pennsylvania. The second course to bear the explicit title of statistics appears to have been one taught at the University of Pennsylvania in 1887 by Professor Roland P. Falkner, who was probably the first man in America to have the title Professor of Statistics.

In the same year, the first psychology course to mention statistical method was taught by Professor James McKeen Cattell, also in the University of Pennsylvania. There appears to be general agreement that Cattell's teaching, both at the University of Pennsylvania and, more especially later at Columbia, combined with his use of statistics in his own writings, was the greatest single factor making for the adoption of statistical methods by American psychologists. From the first, his teaching was permeated by an enthusiasm for measurement and by a knowledge of the technique of treating the results of that measurement, with much talk of probable errors.

The American Journal of Psychology. In November, 1887 appeared the first issue of the *American Journal of Psychology*, in which G. Stanley Hall as editor and founder stated that "The object of this Journal is to record the psychological work of a scientific, as distinct from a speculative character." From that first issue its pages stimulated the statistical tendency in psychological studies, and so, indirectly encouraged the tendency to offer instruction in statistical methods to advanced students of psychology.

Statistics at Clark University. In 1888, Hall was asked to take the presidency of the newly founded Clark University, and was immediately given a year's leave of absence for European travel to select his faculty and to plan the organization of the school. The first classes were opened in 1889. Clark became at once a center for statistical research in anthropology and psychology. Professor Franz Boas was there from 1889 to 1892 lecturing on the applications of statistics to anthropology and directing statistical research in many fields. During these years there appeared several important psychological studies which made extensive use of statistical techniques, the authors referring to Boas as the one who advised them about statistical matters. E. W. Scripture came from Leipzig in January, 1891, and for the rest of that year was at Clark as research fellow in psychology, performing experiments and constructing apparatus to

measure the temperature sense. The next fall he went to Yale as professor of psychology. At the same time, Thaddeus L. Bolton was a scholar in psychology at Clark, making a large scale statistical study of memory in school children under the guidance of Boas. The same year, Gerard M. West delivered lectures on the growth of children, and A. F. Chamberlain made an extensive study of the anthropological measurements of 20,000 school children of Toronto. When Boas left in 1892 to become chief assistant in the department of anthropology at the World's Fair in Chicago, and afterwards director of similar work in the Field Museum, Chamberlain succeeded to the position thus left vacant and continued until his death in 1914 to give a course called anthropological psychology, which seems to have included some work on measurements and the laws of growth. William L. Bryan was also a student at Clark in 1892 and completed his study of voluntary motor ability, the statistical analysis being carried out on lines suggested by Boas.

Other Early Statistics Courses. In 1891, Michigan and Yale each introduced their first courses in statistics. At Michigan, Professor F. C. Hicks taught "Theory of Statistics" in the department of political economy, a course taken over by Professor C. L. Cooley two years later. At Yale, Professor Irving Fisher, also in political economy, gave a course called "The Mathematical Theory of Prices," which seems to have been the forerunner of courses in statistics which he has taught since then. In the fall of 1892, Professor E. W. Scripture joined the faculty of Yale, where he remained until 1903 teaching a psychology which made much use of statistical methods. In 1897 he became director of the Yale Psychological Laboratory. His text *The New Psychology* (1897), with one chapter entitled "Statistics," and another on "Measurement," and with appendices giving a table of values for the probability integral, schemes for Bernoulli's Theorem, and a discussion of the median, average, probable error, mean square error and similar measures, suggests that he was making very considerable use of statistical methods in the courses in psychology which he taught during these years at Yale. Dean Carl E. Seashore, who was one of Scripture's students at Yale, makes the statement[7] that " . . . . next to Cattell, Scripture was the man who first stood strong for statistical method. He was very well trained in mathematics and was far ahead of his time in the application of mathematics to laboratory psychology. Unfortunately, he has not published on the subject, and the thing was so new at the time that

[7] In a letter to the writer, Jan. 28, 1927.

it does not show very much in the studies from the Yale Psychological Laboratory, but if he had stayed in the field, he would have done much to give statistics a place in the psychological laboratory."

Schools Teaching Statistics in 1891. Thus by 1891 at least five schools—Columbia, Pennsylvania, Clark, Michigan, and Yale—provided some form of definite classroom instruction bearing on the theory and practice of statistics. The dates of the first courses in statistics in the nine schools studied are shown in Table I.

TABLE I

First Course in Statistics Offered in Each of 9 Selected Universities

School	Year	Department
Columbia University	1880	Economics
University of Pennsylvania	1887	Economics Psychology
Clark University	1891	Psychology Anthropology
University of Michigan	1891	Economics
Yale University	1891	Economics
University of Chicago	1893	Economics
Harvard University	1895	Economics
University of Illinois	1896	Economics
University of Iowa	1900	Economics

Statistics Courses in 1896. By 1896, possibly ten or twelve universities in America offered instruction of some formal sort in statistical theory and methods. For some years, both Yale and Clark had presented such work in their departments of psychology; while Pennsylvania, Chicago, Columbia, Harvard, Michigan, Yale, and doubtless others had utilized it in the fields of political economy and sociology. In this year, Cattell gave for the first time his course in "Mental Measurement," and as this was given in the department of philosophy, psychology, and education, perhaps it might be considered to be the first course in "Educational Statistics." That designation, however, seems more truly applicable to the course in "Child Study" first given by Thorndike in 1899–1900. Table II shows the names of courses which according to the catalogues, were taught in the twelve selected schools in the year 1896–97. It is not claimed that the list contains all the courses taught in any college in that year. This particular year was chosen for no reason except that some year must be selected and this

TABLE II

College Courses Providing Instruction in Statistics 1896–97

INSTITUTION	INSTRUCTOR	DEPARTMENT	NAME OF COURSE
University of Chicago	E. R. L. Gould	Political Economy	Statistical Investigation
Clark University	A. F. Chamberlain	Psychology	Anthropological Psychology
	E. C. Sanford	Psychology	Laboratory Course of Pedagogical Tests and Measurements
Columbia University	Richmond Mayo-Smith	Economics and Sociology	Statistics and Sociology
	Richmond Mayo-Smith	Economics and Sociology	Statistics and Economics
	Richmond Mayo-Smith	Economics and Sociology	Theory of Statistics
	Richmond Mayo-Smith	Economics and Sociology	Work in the Statistical Laboratory
	J. McK. Cattell	Philosophy, Psychology and Education	Mental Measurement
Harvard University	John Cummings	Economics	Statistics—Theory and Method
Johns Hopkins University	H. L. Moore	Economics	Mathematical Economics
University of Illinois	M. B. Hammond	Economics	Statistics
University of Michigan	Charles L. Cooley	Economics	Theory and Practice of Statistics
	Charles L. Cooley	Economics	Special Studies in Sociology and Statistics
University of Pennsylvania	Roland P. Falkner	Economics	Introduction to Statistics
	Roland P. Falkner	Economics	Statistics of Economic Problems
	Roland P. Falkner	Economics	Statistical Organization
	Roland P. Falkner	Economics	Statistical Verification of Economic Theory
Yale University	Irving Fisher	Economics	Statistics
	E. W. Scripture	Psychology	Psychology—advanced course
	E. W. Scripture	Psychology	Research—Work in Psychology

afforded a contrast by which to make vivid the changes which three decades have brought about.

It should be noticed that the only departments making use of the new study are economics and psychology, with the latter much in the minority. Only two schools, Yale and Columbia, have courses in more than one department. Up to this time mathematics departments had displayed little willingness to befriend the newcomer, a neglect rather noteworthy in view of the warm recognition accorded some of its ancestors by Gauss, Laplace, De Moivre, the Bernoullis, Poisson, and a host of other distinguished mathematicians. Astronomy and engineering had made a polite but distant bow to the subject through courses in "Theory of Probability and Least Squares," but even these were confined chiefly to technical schools. From reading the catalogue descriptions of the courses in economic statistics, one is led to conclude that most of them were stressing statistical facts and compilations

TABLE III

FIRST COLLEGE COURSES IN STATISTICS BY DEPARTMENTS

DEPARTMENT	YEAR	SCHOOL	INSTRUCTOR
Economics	1880	Columbia	Mayo-Smith
Psychology	1887	Pennsylvania	Cattell
Anthropology	1889	Clark	Boas
Biology	1897	Harvard	C. B. Davenport
Mathematics*	1898	Illinois	Myers
Education	1900	Columbia	Thorndike

* This course was entitled "Statistical Adjustments." Courses in probability are, of course, much older.

and the sources where collected data might be obtained, and that they had very little to do with either theory or practice of statistical research.

The first course (in any of the colleges studied) given in departments represented in the six major categories previously mentioned appears to be the one named in Table III.

4. PIONEER COURSES IN VARIOUS DEPARTMENTS

Criteria of Selection. Certain courses in statistical methods stand out as having had an unusual vitality, marked longevity, ability to stimulate research, or as having served as a model for other similar courses. A few of these will now be mentioned briefly.

Psychology. Although Cattell gave no courses with the specific label of statistics, yet his teaching of experimental psychology and mental meas-

urement probably deserve to come under this classification. His instruction in these subjects was indeed noteworthy, not only for its priority as the first psychology courses in America to make consistent and systematic use of statistical methods, but also because in these courses he had as students such men as Thorndike, Woodworth, Wissler, and Ruger, who gave a further impetus to the use of statistics in psychology and education. At Pennsylvania from 1887 to 1891 and at Columbia from 1891 to 1917, he so taught psychology that both psychology and education were permanently influenced. The great awakening of interest in psycho-physical theory in the eighties had produced a volume of data demanding statistical treatment, and so eventually became a force calling for the organization of college instruction, first in statistical method and later in statistical theory. Much of the responsibility for this interest in psycho-physical theory must also be attributed to Cattell.

The study of psycho-physical methods in America owes a great deal to Professor Edward Bradford Titchener, professor of psychology at Cornell University from 1892 to 1927. Nearly all of the books on psychology which he has published at intervals since 1896 contain some discussion of psycho-physical methods, and in the *Experimental Psychology* (1905), II, in both the student's manual and the instructor's manual, the law of error is discussed. A non-technical explanation of the meaning of the law is followed by a statement of its relation to the integral calculus. In the instructor's manual there is also a statement of the author's reasons for including what he terms a "quasi-mathematical discussion." "It is regrettable, on general principles, that any third-year University student, be he mathematically minded or not, should be unable to read such a book, say, as Merriman's Least Squares. It is very regrettable: but it is true, in the author's experience, that the great majority of the students who come into the psychological laboratory cannot. Now these students come into the laboratory for psychology, and for nothing else. They cannot use the methods intelligently without a minimum of mathematical insight; they cannot be taught mathematics—any more than they can be taught neurology or physics—in the time spent upon psychology. What the author has tried to do, therefore, is first and foremost, to treat the methods psychologically; to show in every case where introspection stops and mathematical manipulaton begins, and to hold in view throughout the psychological end to which calculation is the means. Secondly he has tried to say enough about mathematics for the student to realize how necessary an ally mathematics is, when one is working at psychology from the quantitative standpoint, and to understand at least the general trend of the arguments put

forward, e.g., by Fechner and Müller in the classical psycho-physical treatises. . . . . Teachers of psychology have to meet existing conditions and the existing conditions are, all too often, those of teaching quantitative psychology to non-mathematical students. We can insist that our graduate students shall know a certain amount of mathematics; we cannot, as things are, make the same requirement for undergraduates." The list of reference books which the author suggests as helps to the teacher of psychology who might wish to use statistical methods includes the work of such well-known writers as Airy, Bowley, C. B. Davenport, Ebbinghaus, Galton, Jevons, Merriman, Venn, and Whitworth.

Anthropometry. The courses in anthropometry and in anthropological statistics taught by Professor Franz Boas at Clark, 1889–92, and at Columbia since 1896, have been one of the strong influences operating to introduce statistical methods into education. At Clark he inspired and guided numerous important statistical studies in psychology, made at a time when such methods were relatively new in America. Some of these had an important bearing on school problems.

In this connection it may be said that many statistical studies which would now be considered as educational research were in an earlier day often sponsored by boards of health and medical organizations of one kind and another. The activities of Henry Pickering Bowditch of the Harvard Medical School and of such men as William Townsend Porter[8] resulted in some of the earliest American statistical studies related to education. While this is not the chief way in which the influence of Galton reached American education, there is an interesting line of inheritance from Galton and Quetelet through these physicians who made early studies of physical traits, and thus contributed indirectly to American education. Boas was in touch with most of these men, sometimes giving them statistical advice.[9]

Soon after he came to Columbia in 1896, Boas had in his courses Thorndike, Woodworth, and Wissler, who received theoretical and technical training from him at the same time that they were being inspired by Cattell to use the new discipline in the interests of education.

[8] Cf. Bowditch, *The Growth of Children* (Report of the Board of Health of Massachusetts) 1877; and Porter, "The Growth of Saint Louis Children," *Transactions of the Academy of Science of St. Louis*, 1894.

[9] Among the letters of Henry Pickering Bowditch which his son presented to the Genetics Record Office of the Carnegie Institution of Washington, Cold Spring Harbor, Long Island, are two from Boas in the year 1892, discussing certain statistical matters relative to the study of growth. In the same collection is a letter from Galton to Bowditch.

Mathematics. In mathematical statistics, the courses given by Professor James W. Glover at Michigan and by Professor H. L. Rietz at Iowa deserve special notice. From the time Glover began to give his first course in the mathematical theory of statistics until now the mathematics department has had almost a complete monopoly of the work in statistics in the University of Michigan. The work started with the applications of statistics to actuarial theory, and gradually broadened into a group of eight or ten courses taught by Glover and his associates in the mathematics department, and taken by "students in economics and business administration, education, and in sciences dealing with observed, statistical, or experimental data." The department has graduated and placed more than three hundred students in responsible actuarial and statistical positions.

Another notable instance of the absorption of statistics courses by the mathematics department is presented by the work of Rietz, who has been teaching the mathematical theory of statistics, and training statisticians and actuaries, since 1905, in the University of Illinois from 1905–1918, and in the University of Iowa since 1918. The department of economics at Illinois ceased to teach statistics in 1905, printing the titles of the courses given in the mathematics department among their own departmental offerings. His courses are also notable for having had among their members T. L. Kelley, Rugg, Forsyth, and perhaps others who have since become well known as writers upon and teachers of statistics.

Biology. The first university courses in biology to employ statistical methods appear to have been those taught by Professor Charles B. Davenport at Harvard (1887–1899) and at Chicago (1899–1904). Davenport had had mathematical training and held a degree in civil engineering, in which he had had a year's professional experience before taking up biology. His *Statistical Methods in Biological Variation*[10] seems to have been the first text devoted entirely to statistics to appear in America. It was designed as a handbook which a student of biology might use as an engineer uses his manual. This book seems to have been the first to introduce the newer methods of calculating correlation and related measures, which Pearson and his associates had been working out in the preceding two or three years.

For some years a course using statistical methods, and called "The Principles of Evolution as applied to the Improvement of Domesticated Animals and Plants," was taught by Professor Eugene Davenport, head

[10] 1st ed. 1899, 2d ed. 1904, 3rd ed. 1914.

of the department of thremmatology and later Dean of the Agricultural College at the University of Illinois. He was influenced somewhat by the work of Charles B. Davenport, somewhat by Galton's *Natural Inheritance*, and somewhat by close association with Rietz. At that time Davenport was Director of the Agricultural Experiment Station of the University. Needing statistical assistance, he appointed Rietz statistician to the bureau, a position which he retained for many years. Thus Rietz appears to have entered the statistical field by the route of scientific agriculture. In 1907 appeared Davenport's *Principles of Animal Breeding* with a statistical appendix by Rietz. This was in all probability the first systematic use of statistical theory in agricultural education. It was also the first of the series of statistical papers by Rietz. In the preface the author italicizes his assertion that " . . . . nothing is clearer than that *the successful breeder of the future will be a bookkeeper and a statistician*." This course and this book established statistical methods as an essential feature of scientific agriculture, and had a wide influence on the development of agricultural education. The latter half of the book is built around the ideas of correlation and regression, and bears resemblances to *Natural Inheritance*, from which it quotes at length, adding applications and problems taken from the field of animal husbandry.

The course in "Variation, Heredity, and the Principles of Animal Breeding" organized at Harvard by Professor W. E. Castle about 1908, and the course in "Statistical Zoölogy" which Professor Frank Smith began to teach at the University of Illinois in 1902, should be mentioned among early courses in biological statistics.

Economics. Most of the courses just mentioned have produced some discernible effect on the trend of educational statistics, either directly because educational research was stimulated and guided, or indirectly because some student in the course later entered education as a profession, or because from the classroom lectures there developed a text which has been useful to students of education. The connection between the teaching of economics and educational statistics is not so easily traced until very recently and contributions of one to the other appear relatively slight.

Falkner's work at the University of Pennsylvania has already been mentioned as one of the earliest courses to bear the appellation "Statistics." While this particular course may have had no pronounced effect on the scientific study of education, certainly Falkner himself did have a considerable influence, first through his position as Commissioner of Education in Porto Rico during the Roosevelt administration and then through his

association with Ayres.[11] There are about a score of other men in the country whose teaching of statistics in connection with the social sciences has been noteworthy and has extended over a considerable period of time, but the writer has not been able to discover any close connection between this teaching and the application of statistical method to educational problems. At the present time, it is true, the use of statistics by men working in educational administration is often concerned with problems which are truly economic in their nature, but this is a recent development, and should be considered an outcome rather than a contributing factor in the statistical movement in professional education.

Education. The first course which may properly be designated as "educational statistics" was the one given by Professor E. L. Thorndike in Teachers College of Columbia University in 1899–1900, under the title of "Child Study." The subject matter of this course appeared in book form in June 1901, as *Notes on Child Study*, with a second edition two years later. The cleavage between Thorndike's methods and the traditional manner of studying children through a collection of anecdotes was complete. The statistical devices employed in this text are of the simplest, and include little more than histograms, averages, mean variation, frequency tables, frequency curves, overlapping frequency curves showing the percentage of one group that reach or exceed the median of another group, and the use of statements of numerical probability in a very simple form. The type of explanations which the author feels it necessary to make indicates how utterly foreign these concepts were to students of education at the beginning of this century. The book is replete with warnings against hasty generalizations from mass data and with suggestions as to the need of collecting statistics and making exhaustive studies to acquire new information about the prevalence of defects or the relation of mental and developmental defects, and the like. "To be properly studied, growth must be studied on a large scale and with great care, and no teacher should indulge in general theories on the basis of slight experience." Reading this, we recall that Thorndike had just been studying with Boas, who for over ten years had been teaching anthropology with a strong emphasis on statistics. In the first chapter, Thorndike combats "the nonsensical idea that because a certain statement about a group of facts is not true of every individual in the group, it is worthless," and, in the most elementary fashion possible develops the idea of averages and variations, terms which were evidently not a part of the pedagogical vocabulary of that day. The book, which is probably a

[11] See page 166.

reflection of the spirit of the course, was intended as a text in Child Study, not as a manual to guide in the technique of research. No instructions were given for computation and no suggestions as to where such instructions could be found, but there was a tacit assumption that instructors who used the book might be familiar with such methods. The book breathes an infectious air of inquiry. An eager-minded student reading it must have come away with a desire to go out and measure *something* and to submit the results of his measurement to quantitative analysis. Lectures of this sort would soon create a demand for a manual of statistics designed particularly for students of education. Thus Thorndike introduced the methods of the older sciences into the field of education.

In 1902, Thorndike delegated the teaching of this course to Professor Naomi Norsworthy and organized his more widely known course in "The Application of Psychology and Statistical Methods to Education." This he continued to teach until 1920, being assisted in 1919 by McCall to whom he turned over the course in 1921. The need for a text produced the first edition of the *Introduction to Mental and Social Measurements*[12] in 1904.

While *Mental and Social Measurements* was still but a type-written syllabus, there were in this course several students whose interest in statistical methods of research has been an important factor making for the widespread use of statistics by schoolmen. In 1903–04 the twelve members of the class included Dearborn, Elliott, Henmon, Ruger, Strayer, and Suzzallo, and the next year Cubberley and Van Denburg were in the class. Several of these were studying with Cattell at the same time.

One of the first dissertations to be published by a member of these groups was *City School Expenditures, the Variability and Interrelations of the Principle Items*, by Strayer. This study was a pioneer in its field, the first large-scale statistical study to be made by a student of education in a professional school of education for the purpose of determining a policy of educational administration. The manner of arriving at conclusions by means of the coefficient of correlation, and by computations of Q, $\frac{2Q}{\text{Median}}$, $\frac{2Q}{\sqrt{\text{Median}}}$, and other similar expressions, was in 1905 so unexpected, so alien to the orthodox educational vocabulary, that Thorndike wrote an introduction to reassure the reader. To one whose chief concern is in the historical development of statistics rather than in the administrative principles proposed, this introduction is not the least interesting feature of the book. A vivid description of the type of educational study ordinarily made at that time is implied in Thorndike's comments

[12] Second edition, 1913.

on this one. "The methods being in some respects new in the literature of education, deserve some comment. It is impossible to gain adequate insight into facts as complex as these of school expenses and school achievements without some use of technical statistical methods. If Mr. Strayer's report puzzles some readers by its tables of frequency, its constant use of the median rather than the average as a measure of central tendency, its coefficients of correlation and their corrections, it is of necessity. The facts could not otherwise be handled properly—in some cases not at all. The one matter of technique which needs some explanation and perhaps apology is the use of the Pearson Coefficient of Correlation. This . . . . is indispensable but necessarily obscure. . . . . On the whole, although the arithmetic labor of computing these coefficients is enormous, they should be used in all studies of mental and social traits. . . . . One other feature of Mr. Strayer's method of presentation needs comment—his careful arrangement of the individual measures from which his later results are derived. . . . . The reader who is irritated, as well as awed, by the pages of individual records must remember that in the social sciences lumping facts into averages and totals conceals far more truth than it reveals, and destroys half the value of the record to the expert." This study had much to do with establishing the statistical type of research in educational administration.

5. OUTLINE OF THE DEVELOPMENT OF THE TEACHING OF EDUCATIONAL STATISTICS IN AMERICA

Lines of Influence. The work of Thorndike has been discussed in some detail because it represents the most immediate single cause for the development of the statistical trend in scientific studies of education in America. There have been, however, several other independent lines of influence which should not be overlooked. We will now look briefly at these various lines of inheritance.

European Influence. The two early sources of statistical stimulation which seem to have had the greatest effect upon American education are the work of Francis Galton and the work of the American students who studied in Germany in the eighties. The work of Galton's successor, Pearson, and his associates, continues to the present time to be the dominant influence in shaping educational statistical theory. The work of the Scandinavian school—Opperman, Gram, Thiele, Westergaard, Charlier, and others—is not generally well-known in this country and can scarcely be said to have had an important formative effect upon the theory or the

practice of educational statistics. The works of the Russian writers Markov,[13] Tchebycheff,[14] and Tchuproff[15] or of the many Italian statisticians, are even less familiar. An examination of the sources quoted by any representative sample of recent American texts will bear out these statements.

The Influence of Leipzig. Some of those Americans who studied in Germany in the eighties and early nineties were largely responsible for the introduction of quantitative and statistical methods into American education. The great number of German books on statistics and on the theory of errors of observation which appeared during the latter half of the nineteenth century, as well as the fact that it was in Germany that the psychophysical movement had its inception, must be kept in mind. There was at that time probably a larger literature on these subjects in German than in any other language. The first measurement of a reaction time was made in Germany.[16] The influence of Lexis had been widespread. The use which Ebbinghaus made of quantitative methods to study psychological processes[17] set the model for similar investigations. Fechner's studies[18] in

[13] A. A. Markov was born in 1856. Isserlis (?) speaks of his death as having been hastened by privations after the Russian revolution, but the writer has been unable to find any biographical account. He was a brilliant writer on the theory of probabilities and mathematical statistics.

[14] Pafnutii Lvovitch Tchebycheff (1821–1894) was one of Russia's most celebrated and versatile mathematicians, a Professor of Mathematics at the University of St. Petersburg, where he taught a wide variety of subjects. Translations of his work on the distribution of prime numbers and of his theorem on mean values are included in the forthcoming *Source Book in Mathematics*. For further biographical details, the reader is referred to a brochure by A. Vasilief entitled *P. L. Tchébychef et son Œuvre Scientifique* (Turin, 1898) or to a sketch by C. A. Possé in the *Dictionnaire des écrivans et savants russes rédigé par M. Vénguerof*, reprinted in Vol. II of Markov's edition of Tchebycheff's *Œuvres*.

[15] Alexander Alexandrovitch Tchuproff (1874–1926) was a scholarly and brilliant writer on mathematical statistics whose life work was the building up of sound logical foundations for theoretical statistics. After the Russian revolution he lived in exile and suffered much privation. Most of the work he did in these years is yet unpublished. A biography signed "L. I." (presumably Isserlis) may be found in the *Journal of the Royal Statistical Society*, LXXXIX (1926), 619–622.

[16] See page 24.

[17] "Über das Gedächtnis (Leipzig, 1885). English translation by Ruger (1913).

[18] Among these were the following:

"Ueber die Bestimmung des Wahrscheinlichen Fehlers eines Beobachtungsmittels durch die Summe der Einfachen Abweichungen."

Element der Psychophysik (1860, 1889, 1907).

Zur experimentalen Aesthetik (1871).

In Sachen der Psychophysik (1877).

psycho-physics and statistics were widely read and powerful in their influence. Until his death in 1887, Fechner lived and wrote in Leipzig. Also teaching at the University of Leipzig were the physiologists Ludwig[19] and Weber,[20] both of whom made noteworthy use of quantitative methods of research. While many studies of reaction time came from Wundt's laboratory, it does not appear that Wundt himself was committed to a belief in the statistical treatment of the results of experimentation. Among the American students of this era who returned from Germany with a strong inclination toward the quantitative treatment of variable phenomena, only the most outstanding will be mentioned.

Henry Pickering Bowditch studied physiology under Ludwig.[21] Returning to America in 1871 he established at Harvard the first physiological laboratory in America for the use of students. Here he made studies of reaction time, of the indefatigibility of the nerves, the effects of different rates of stimulation on the action of nerves, and of the height of school children. Here Hall and Southard performed psychological experiments, and students in various other fields made quantitative investigations of a sort that had not been previously possible. The study of the growth of school children which Bowditch published in 1877 was one of the first large-scale statistical studies made in America of a problem closely related to the work of the public schools.

Mention has already been made of the work of Boas, Cattell, Titchener, Falkner, and Scripture, all of whom studied in Germany, and all of whom began to teach in this country between the years 1887 and 1892, after a period of study in Germany. During his year's teaching in Oxford, Cattell had been associated also with Galton.

The Teachers College Line of Influence. The main channel by which the stream of statistical influence has reached the field of education has been Teachers College, Columbia, or more particularly Thorndike. Upon him the influence of Boas and Cattell converged, and from him a similar influence reached out to all phases of professional education. As a part of this influ-

"Ueber den Ausgangswerth der kleinsten Abweichungssumme, dessen Bestimmung, Verwendung, und Verallgemeinerungen" (1878).

Revision der Hauptpunkte der Psychophysik (1882).

Collectivemasslehre (published posthumously (1897).

[19] Karl Friedrich Wilhelm Ludwig (1816–1895). For a biography see the *Proceedings of the Royal Society*, XLVII.

[20] Ernst Heinrich Weber.

[21] Also under Claude Bernard in Paris.

ence might be counted the teaching of statistics by Professor Truman L. Kelley (University of Illinois summer of 1909; Teachers College, summer of 1914; University of Texas, 1914–17; Teachers College, 1918–20; Stanford, since 1920), by Professor Henry A. Ruger (University of Nebraska, summer course; Teachers College, since 1911), by Professor William A. McCall (Teachers College, since 1916), and by the many teachers of statistics in other schools of education who have studied under these men. Here also should be counted the emphasis placed on statistical methods in the courses presented at Teachers College for superintendents of schools. Since Professor George D. Strayer began to teach there in 1910, the courses designed for superintendents of schools have included definite training in the simpler aspects of statistics, and a very large proportion of the dissertations written in the field of administration are statistical in nature. The effect of this upon the country at large has come both through city superintendents of schools who have made use of elementary statistics in their local administration, and through men trained at Teachers College who are giving courses in other schools of education. Another effect traceable to the same source is the extensive use in education of scales, the development of a technique of constructing them, and a large part of the application of statistical methods in educational measurement. Of the men and women who have constructed well-known educational scales, the majority have been students of Thorndike. Any history of the construction and use of educational tests and scales would necessarily recognize Thorndike as the central figure in the movement, a leader in developing techniques of making scales and in utilising them for the study of the problems of the American school. A careful history of this movement, taking into account its theoretical background in Galton's *grades and deviates*, in his studies of first and second prizes, in some of Edgeworth's studies, in Pearson's *mentaces*, and particularly in the Weber-Fechner law, might well become an extended treatise, and such is not included in this present study.

The Clark University Line of Influence. It has already been said that important statistical studies of educational problems were made at Clark University under the direction of Boas as early as 1890. A number of men who studied with him at that time have since had considerable influence upon the trend of professional education.

Falkner and Ayres. Another line of development in educational statistics which is almost entirely independent of Teachers College came through Falkner and Ayres. Ayres was a superintendent of schools in

Porto Rico in 1902–07, and was closely associated with Falkner, who was then the commissioner of education in the Islands. The latter had previously taught statistics in the University of Pennsylvania. Together they were active in experimental studies of education in the Islands at the same time that Thorndike and some of his students were making similar studies in the United States. Returning to this country in 1908, Ayres worked on a study of backward children which was being conducted by the Russell Sage Foundation, and later became head of the division of education in the foundation, where he conducted many studies that had a strong leaning toward economic research.[22] His study of retardation in the schools had a wide influence on the thinking of school men throughout the country. During the decade before the World War, when statistical work in education was developing very rapidly, Ayres was an active participant in the movement, a fairly regular attendant at educational conventions, writer of several books and a very large number of monographs on educational problems, lecturer and teacher of summer courses in several universities. His work on spelling scales constitutes one of the early chapters in the development of standardized tests and scales in this country. He did not use complicated formulas, but made his conclusions convincing to the layman by means of very simple statistical devices.[23] Thus he had a

[22] Among these may be noted:

Laggards in Our Schools: a Study of Retardation and Elimination in City Systems, 1909.

Physical Defects and School Progress, 1909.

Open-air Schools, 1910.

Relative Responsibility of School and Society for the Over Age Child, 1910.

Identification of the Misfit Child, 1911.

Provision for Exceptional Children in Public Schools, (Bulletin of the U. S. Bureau of Education) 1911.

New Attitude of the School towards the Health of the Child, 1911.

Measurement of Educational Processes and Products, 1912.

Effect of Promotion Rates on School Efficiency, 1913.

Constant and Variable Occupations and their Bearing on Problem of Vocational Education, 1914.

Public Schools of Springfield, Illinois, 1914.

Measuring Scale for Ability in Spelling, 1915.

Spelling Vocabularies of Personal and Business Letters, 1915.

Health Work in the Public Schools, (Cleveland Educational Survey), 1915.

Psychological Tests in Vocational Guidance, 1916.

School Organization and Administration, Cleveland (Cleveland Educational Survey), 1916.

Index Number for State School System, 1920

Many other similar studies.

[23] If a section on short cuts and computational devices were included in this study, Ayres's name would appear in it, although his work in this connection is less extensive than the later work of Thurstone, Otis, and the still more important works of Toops.

considerable influence in making statistical methods popular among students of educational administration and finance, particularly among those who had had no technical training in the subject. His influence upon the course of development of educational statistics seems to have been quite out of proportion to the amount of direct teaching of the subject which he did.

Chicago University. In considering the teaching of statistics in schools of education, we are led to notice the difference in the type of research studies that have issued from the School of Education of the University of Chicago and from Teachers College of Columbia. More and more the latter have tended to be of a statistical nature, while among the former, laboratory investigations rather than mass measurement have predominated. Courses in statistics in the School of Education of the University of Chicago date from about 1913–14, in which year Professor Franklin Bobbitt gave a course entitled "Statistical Method as Applied to Educational Problems," and Professors Judd and Freeman gave another called "Experimental and Statistical Problems in Education." Professor Harold Rugg joined the faculty of the school in 1915, and taught statistics there until 1920 when he left Chicago for Columbia. His *Statistical Methods Applied to Education* appeared in 1917 and accelerated the already strong movement for "the quantitative study of school problems." The author had studied mathematical statistics with Rietz and had been in close touch with Ayres, and the book indicates knowledge of the theoretical background of statistics utilised for the solution of practical problems. It brought within the reach of students of education some aspects of statistical theory which at that time were available only in the original memoirs. It also contained a considerable bibliography of quantitative studies of education and several tables for the facilitation of computation.

Pearson. The influence of Pearson upon educational statistics in America has been so profound that it is all but impossible to overstate the extent to which he is responsible for modern statistical theory. In the first place, almost all of the material taught in courses in statistical method, whether educational or otherwise, had its source in his writings or those of his associates and students. A blanket scheme of organization applicable to the history of almost every major topic employed in educational statistics might be arranged under three heads: (a) Minor work of writers before Pearson, (b) Establishment of the theory by Pearson and his associates, (c) The applications of the theory to educa-

tional problems by various writers. When a recent American textbook states that "the Pearson product moment is not the best method of computing correlation," it is worthy of note that the method recommended by the author as an improvement upon Pearson's is merely a computational device used by Pearson for many years, and published by him in 1896 and again in 1907. It is probably true that the criterion on the basis of which material has been selected from his writings for use in education has not always been its pertinence, but often chiefly its simplicity, and that whatever can be reduced to relatively simple rules has been adapted to the uses of education without over much attention to the hypotheses from which the rules were derived. Nevertheless it remains true that the great body of our statistical theory is derived either directly or indirectly from the writings of the Pearson school. The writer believes that those American educators who have themselves done the most work in statistical theory are exactly the ones who would agree most emphatically with this statement. Disagreement is more likely to come from those who know only the secondary sources. Pearson's series of masterly memoirs on the mathematical theory of evolution, and his work as editor of the quarterly journal *Biometrika* make him indeed comparable to the creators of new sciences.

In the second place, he has had direct personal contact with a great many men who are now teaching statistics in America. A list of all these is not available, but within the field of education alone, at least three men who are now teaching statistics—Holzinger, Kelley, and Ruger—have been associated with him in statistical work in the Galton Eugenics Laboratory in University College, London.

Other Stimuli. These seem to have been the major lines of influence in developing the theory and practice of statistics in education. It is scarcely possible—even if desirable—to report here all the individual and separate forces which have contributed to the present status of educational statistics. The field is new and most of the men who are at the present time doing the most to extend it have been teaching for so short a time that it would be unfair to them to attempt at this time to enumerate or to evaluate their contributions. An appraisal of the work of men now actively engaged in teaching, even if accurate today, would almost certainly be inaccurate tomorrow. The omission of the work of men who are giving or have recently given valuable courses in educational statistics makes the picture of the advance in the teaching of that subject somewhat misleading. Since no *tertium quid* could be discovered, the second horn of the dilemma has been selected as appearing slightly less perilous.

6. THE PRESENT SITUATION

Probable Extent of College Instruction in Statistics. The number of college courses in statistics which are being taught at the present time is difficult to ascertain. About one hundred and twenty-five institutions returned in usable form the questionnaire blanks sent out in 1925 by the Committee on Educational and Professional Standards of the American Statistical Association. Of these, forty schools reported that they had no courses in statistics, although several announced their intention to introduce such a course the following year. Five schools reported that statistics was handled incidentally in courses in education and psychology, but not in courses devoted solely to statistics. Eighty-four schools reported two hundred and twenty-two courses in mathematics, psychology, education, or in unspecified departments that appeared to be related to these. Thirty-

TABLE IV

Distribution of Statistics Courses by Departments in 84 American Colleges and Universities, 1925

Department	Elementary Statistics	Advanced Statistics	Related Courses
Mathematics	57	40	35
Economics and Social Sciences	86	8	7
Schools of Business	53	3	15
Education	34	10	28
Psychology	12		
Public Health	11	3	
Agriculture	4		

three schools reported courses in departments other than these three. In all probability these figures are much too low. The writer has personal knowledge of the work of two institutions not included in this list, in which there were at that time five statistics courses in education, one in mathematics, and one in economics. Glover's figures[24] taken from the replies to this same questionnaire show the distribution of courses indicated in Table IV.

Diversity of Practice. One point at least is clearly demonstrated by the questionnaire returns. There is wide diversity of practice in the content of courses called statistics, in the length of time the subject is pursued and the

[24] "Statistical Teaching in American Colleges and Universities," *Journal of the American Statistical Association*, XXI (1926), 419–424.

number of hours a week allotted to it, in the maturity of students enrolled, in the provision of laboratory equipment, and most of all in the mathematics prerequisites specified. It appears that "advanced statistics" has a very wide variety of meanings, and may have prerequisites in mathematics ranging all the way from integral calculus down to nothing at all. Such confusion is probably to be expected in a subject which has so recently entered the university field and which has not yet secured any considerable uniformity of language or of symbolism, and certainly little uniformity of teaching aims. The instructors of these courses probably include at one extreme the man who believes with Fourier that statistics will make progress only as it is retained in the hands of those versed in the higher mathematics and who would therefore exclude all others from studying it, and at the other extreme the man who feels that the type of thinking called for in the study of mathematics is incompatible with the aims of sociological and educational research. We are still in the realm of personal opinion—expert and conflicting. A philosophy of the teaching of statistics is still far to seek.

In so far as this condition indicates the virility and growth of a young subject, it is an asset. However, it would seem that university courses in statistics, even in educational statistics, which is the youngest member of the family, have attained a maturity which would make profitable an attempt to secure some consensus of opinion on questions such as the following: Shall we continue the practice of allowing each department in a school to present its own courses in statistics? Shall we follow the example of Michigan, Illinois, and Iowa and delegate to the mathematics department the responsibility for teaching statistics? Shall we urge that different departments unite in a single elementary course presenting the general principles of statistics, applicable to any type of inquiry, and that they reserve for advanced courses the aspects of statistical theory or practice peculiar to their own particular needs? Shall we adopt Professor Chaddock's suggestion that statistics should be required of all undergraduates majoring in the social sciences just as laboratory work is required of students of the physical sciences? Is it possible to determine the minimum amount of mathematical training necessary for success in learning to read articles that employ statistical language, in learning to make statistical computations under guidance, in planning statistical investigations, in creating new statistical theory? Is it feasible to plan a differentiation of instruction that will take account of different degrees of preparation and different purposes of study? How long a period of instruction is needed to accomplish each of these ends? Is it desirable that all students be given an "appreciation course" designed to teach them to read the new language which scientific

studies in all the social sciences increasingly employ? If so, is there some way to discourage these prospective consumers of statistics from becoming producers of statistics without further training? Without waiting for the sanction of an international statistical congress, is it possible to secure enough uniformity in terminology and in symbolism so that students will be able to pass from one school to another, from one instructor to another, without being obliged to learn an entirely new language, and so that students will be encouraged to read the works of a wider range of statisticians?

7. REMARKS BY FLORENCE NIGHTINGALE CONCERNING THE NEED OF COLLEGE INSTRUCTION IN STATISTICS

Very few universities in American are now without one or more courses in statistics, and specialized training may be obtained in most statistical fields. However, in spite of all this attention to the teaching of statistics, it is impossible, even today to read the letter written by Florence Nightingale to Francis Galton in 1891 on the subject of a university professorship in statistics, without a vivid sense of its stirring challenge. The letter is quoted in full in Pearson's *Life, Letters, and Labours of Francis Galton*, II, 416–418, and we will reproduce only a portion of it here.

"10 South Street, Park Lane, W. *Feb.* 7, '91.

Scheme of Social Physics Teaching

DEAR SIR, Sir Douglas Galton has given me your most kind message; saying that if I will explain in writing to you what I think needs doing, you will be so good as to give it the experienced attention without which it would be worthless. By your kind leave, it is this:

"A scheme from someone of high authority as to what should be the work and subjects in teaching Social Physics and their practical application in the event of our being able to obtain a Statistical Professorship or Readership at the University of Oxford.

"I am not thinking so much of Hygiene and Sanitary work, because these and their statistics have been more closely studied in England than probably any other branch of statistics, though much remains to be desired: e.g. the result of the food and cooking of the poor as seen in the children of the Infant Schools and those of somewhat higher ages. But I would,—subject always to your criticism and only for the sake of illustration—mention a few of the other branches in which we appear hardly to know anything, e.g.

"*A*. The results of Forster's Act, now 20 years old. We sweep annually

into our Elementary Schools hundreds of thousands of children, spending millions of money. Do we know:

"(i) What proportion of children forget their whole education after leaving school; whether *all* they have been taught is *waste?*

"(ii) What are the results upon the lives and conduct of children in after life who don't forget all they have been taught?

"(iii) What are the methods and what are the results, for example in Night Schools and Secondary Schools, in preventing primary education from being a *waste*?

"If we know not what are the effects upon our national life of Forster's Act is not this a strange gap in reasonable England's knowledge?

"*B* (1) The results of legal punishment—i.e. the deterrent or encouraging effects upon crime of being in gaol. Some excellent and hard-working reformers tell us: Whatever you do, keep a boy out of gaol—work the First Offender's Act—once in gaol, always in gaol—gaol is the cradle of crime. Other equally zealous and active reformers say—a boy must be in gaol once at least to learn its hardships before he can be rescued. Is it not again strange in practical England that we know no more about this?

"*B* (2) Is the career of a criminal from his first committal—and for what action—to his last, whether (a) to the gallows, or (b) to rehabilitation, recorded? . . . .

"In how many cases must all our legislation be experiment, not experience! Any experience must be thrown away.

"*B* (3) What effect has education on crime?

"(a) Some people answer unhesitatingly: As education increases, crime decreases. (b) Others as unhesitatingly: Education only teaches to escape conviction, or to steal better when released. (c) Others again: Education has nothing to do with it either way.

"*C*. We spend millions in rates in putting people into Workhouses, and millions in charity in taking them out. What is the proportion of names which from generation to generation appear the same in Workhouse records? What is the proportion of children de-pauperised or pauperised by the Workhouse? . . . . Upon all such subjects how should the use of statistics be taught? . . . . I have no time to make my letter any shorter, although these are but a very few instances. What is wanted is that so high an authority as Mr. Francis Galton should jot down other great branches upon which he would wish for statistics, and for *some teaching how to use these statistics in order to legislate for and to administer our national life* with more precision and experience. . . . . One of the last words Dr. Farr of the General Register Office said to me was: "Yes, you must get an

Oxford Professorship; don't let it drop." . . . . Might I ask from your kindness—if not deterred by this long scrawl—for your answer in writing as to heads of subjects for the scheme? Then to give me some little time, and that you would then make an appointment some afternoon, as you kindly proposed, to talk it over, to teach, and to advise me? Pray believe me, Yours most faithfully, FLORENCE NIGHTINGALE."

Upon this letter Pearson comments: "He was then 69, and she was 70. She was reviving one of the great dreams of her younger days, and he with no sign yet of age, was then actively contributing not a little towards its realisation. . . . . I confess—but then I am a prejudiced person, for the prophetess was proclaiming my own creed—that this letter appears to me one of the finest that Florence Nightingale ever wrote. What is more it is almost as true to-day as it was thirty years ago. We are only just beginning to study social problems—medical, educational, commercial—by adequate statistical methods, and that study has at present done very little to influence legislation. What is more the requisite statistical teaching on which real knowledge must be based has hardly yet spread throughout our universities. The time has yet to come, when the want of a chair of statistical theory and practice in any great university will be considered as much an anomaly as the absence of a chair of mathematics. The logic of the former is as fundamental in all branches of scientific inquiry as the symbolic analysis of the latter."

Sir Francis Galton in 1909, aged 87, with Professor Karl Pearson
By permission of Professor Pearson and Professor H. A. Ruger

CHAPTER VIII

THE ORIGIN OF CERTAIN TECHNICAL TERMS USED IN STATISTICS

It is proposed in this section to present a selected list of the technical terms which are likely to be encountered by the student of educational statistics, the name of the person who seems first to have used the term when that can be ascertained, and a reference to the place of its first publication. In some cases a brief explanatory comment has been appended. This has seemed necessary, for example, when the idea was suggested by one man and the term by another, or when someone has given a name to a concept already well known. In a few cases the writer thinks it possible that the term was in use before it was employed by the person named. Under these circumstances, the symbol * has been placed before the name of the term. In a few other cases, the writer has no uncertainty as to the identity of the person who invented the term, but cannot assert with confidence that the work quoted is in reality the first one in which the term was published. In such cases, the date has been omitted from the first line, and the symbol † placed before the name of the term.

It is cause for regret that symbols, as well as terms, could not be included in the list, but that must be reserved for a later work. A few symbols such as *r*, *b*, *beta*, *sigma*, and *eta*, which by frequency of oral usage, have taken upon themselves the nature of words, have been included in the list.

* *Accomplishment Quotient* (*A. Q.*)............................Franzen

"The Accomplishment Quotient, a School Mark in Terms of Individual Capacity," *Teachers College Record*, XXI (1920), 436.

Accomplishment Ratio...................................Franzen, 1922

The Accomplishment Ratio, Teachers College, Columbia University Contributions to Education, No. 125, 1922.

Alienation, coefficient of...................................Kelley, 1919

Galton gave the formula $f = \sqrt{1 - r^2}$ and explained its significance in his paper "Co-relations and Their Measurement, chiefly from Anthropometric Data," *Proceedings of the Royal Society*, XLV (1888–89), 135–145. The value $\sqrt{1 - r^2}$ has been used by nearly every writer on correlation since that time. See page 104 of this work. Kelley called this the *coefficient of alienation* and used the symbol *k* to desig-

* Term may have been used by other persons before the one named.

† The writer named may have used the term in an earlier work than the one named here.

nate it, in "Principles Underlying the Classification of Men," *Journal of Applied Psychology*, III (1919), 50–67.

Allokurtic. . . . . .Pearson, 1905

See *isokurtic*.

Anomic. . . . . .Pearson, 1905

See *nomic*.

Arithmetic mean.

See *mean*.

† *Array*. . . . . .Pearson, 1895

"Mathematical Contributions to the Theory of Evolution.—III. Regression, Heredity, and Panmixia," *Philosophical Transactions*, A, CLXXXVII (1896), 260.

Attenuation. . . . . .Spearman, 1904

"The Proof and Measurement of Association between Two Things," *American Journal of Psychology*, XV.

Average.

The *Oxford Dictionary* says that this term first appeared about 1500, the earliest instances being in connection with maritime trade in the Mediterranean. The derivation is uncertain. The dictionary gives a list of successive meanings of the term, as follows:

(1) Originally an impost on goods. *Obs.*
(2) Any charge in excess of the freight charges, payable by the owner of the goods.
(3) Loss to the owners from damage at sea.
(4) Equitable distribution of such expense, when shared by a group.
(5) Distribution of the aggregate inequalities of a series of things.
(6) The arithmetic mean so obtained, the medium amount, the "common run."[3]

b (for regression coefficient).

For the history of the term, see pages 104, 106, 111.

Bernoullian series.

See *Poisson series*.

Beta.

The functions $\beta_1 = \dfrac{\mu_3{}^2}{\mu_2{}^3}$ and $\beta_2 = \dfrac{\mu_4}{\mu_2{}^2}$ as aids in curve fitting were defined by Pearson in 1895. "Contributions to the Mathematical Theory of Evolution.—II. Skew Variation in Homogeneous Material," *Philosophical Transactions*, A, CLXXXVI, Part I, 351. The higher betas were introduced by him somewhat later.

Beta.

The symbol $b_{12\cdot 34\cdots n}$ for the partial regression coefficient of x_1 on x_2, all other variables being held constant, is in fairly general usage, at least among statisticians who follow the notation of the English school. Kelley has suggested (*Statistical Method*, p. 281) the use of $\beta_{12\cdot 34\cdots\cdot n}$ in the particular case where each variable is expressed as a multiple of its own standard deviation.

Bi-serial r. . . . . .Pearson, 1909

"On a New Method of Determining Correlation between a measured Character *A*,

and a Character B of which only the Percentage of cases wherein B exceeds (or falls short of) a given Intensity is recorded for each grade of A," *Biometrika*, VII (1909).

B-Scale. . . . . McCall, 1923

How to Experiment in Education, 102–109.

C. A.

See I. Q.

Centesimal grades. . . . . Galton

The idea of percentiles, or centesimal grades, is foreshadowed in *Hereditary Genius* (1869), and is more clearly stated in "Statistics by Intercomparison," *Philosophical Magazine*, 4th series, XLIX (1875). The term *centesimal grades* occurs in *Inquiries into Human Faculty and its Development* (1883), and perhaps earlier. See Chapter IV on "Percentiles" for a more complete history.

† *Charlier check*. . . . . Charlier

The method is described by Charlier in "Researches into the Theory of Probability," *Acta Universitatis Lundensis*, I (1905), and may have been printed elsewhere earlier.

Chi-square (*test of goodness of fit*). . . . . Pearson, 1900

"On the Criterion that a given System of Deviations from the Probable in the Case of a Correlated System of Variables is such that it can be reasonably supposed to have arisen from Random Sampling," *Philosophical Magazine*, 5th series, L (1900).

Clitic curve. . . . . Pearson, 1905

"The curve in which the skewness of the array is plotted to its position is termed the *clitic curve*." "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 52.

Coefficient of association. . . . . Yule, 1900

"On the Association of Attributes in Statistics: with Illustrations from the Material of the Childhood Society &c," *Philosophical Transactions*, A, CXCIV (1900), 271, 272. The coefficient of association suggested here is written as

$$Q = \frac{(AB)\,(\alpha\beta) - (A\beta)\,(\alpha B)}{(AB)\,(\alpha\beta) + (A\beta)\,(\alpha B)}$$

Later Pearson wrote it

$$Q = \frac{ad - bc}{ad + bc}$$

Coefficient of colligation. . . . . Yule, 1912

"On the Methods of Measuring Association between Two Attributes," *Journal of the Royal Statistical Society*, LXXV (1912), 579–642.

Coefficient of contingency. . . . . Pearson, 1904

"With a view of lessening the number of coefficients in use, I adopt the following convention: Any expression or function of either the mean square contingency (ϕ^2) or the mean contingency (ψ) (or indeed of any other measure of the contingency),

which, when the grouping is sufficiently small, is theoretically equal to the coefficient of correlation—on the hypothesis of normal frequency—shall be termed a coefficient of contingency." "Mathematical Contributions to the Theory of Evolution.—XIII. On Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series*, I (1904), 9.

Coefficient of correlation.

For a history of the concept see pages 103–106. The term *correlation* is due to Galton. See correlation. The term *coefficient of correlation* was used by Edgeworth in 1892, in his paper "On Correlated Averages," *Philosophical Magazine*, 5th series, XXXIV, 190–204. This is probably its first use.

† *Coefficient of disturbancy*.................................. Charlier

Vorlesungen über die Grundzüge der Mathematischen Statistik, (2nd ed. 1900), and probably earlier.

Coefficient of heredity........................ Eugene Davenport, 1907

Principles of Breeding.

Coefficient of reliability.............................. Spearman, 1910

See *reliability coefficient.*

Coefficient of variation................................ Pearson, 1895

Pearson's definition of the term on page 271 of "Regression, Heredity, and Panmixia," *Philosophical Transactions*, A, CLXXXVII (1896), is $\frac{100\,\sigma}{\text{Mean}}$. Thorndike (*Empirical Studies in the Theory of Measurement*) used the same term for the fraction $\frac{100\,\sigma}{\sqrt{\text{Mean}}}$. It should be noted that the first expression is, like r, a pure number, the order of the moments being the same in the numerator and the denominator. Pearson has called in question the use of the second expression on the ground that it does not meet this qualification.

Contingency grade..................................... Pearson, 1904

"Mathematical Contributions to the Theory of Evolution.—XIII. On Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series*, I (1904), 6.

Conjugate betas.. Kelley, 1923

Statistical Method, p. 282.

Correlation.. Galton, 1888

"Co-relations and their Measurement, chiefly from Anthropometric Data," *Proceedings of the Royal Society*, XLV (1888–89), 135–145. "The statures of kinsmen are co-related variables; thus, the stature of the father is correlated to that of the adult son, and the stature of the adult son to that of the father . . . ." p. 143. For further history, see pp. 92–111.

Correlation ratio...................................... Pearson, 1905

The properties of this measure and the symbol were briefly noted by Pearson in a footnote on p. 303, 304, of an article in the *Proceedings of the Royal Society*, LXXI (1903). A fuller discussion and the term *correlation ratio* were given in 1905,

at which time Pearson said, "It has been systematically used in my laboratory for some years, and determined longside r for many distributions." "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression." *Drapers' Company Research Memoirs*, II (1905).

Correspondence by rank....................................Otis, 1916

"Some Logical Aspects of the Binet Scale," *Psychological Review*, XXIII, 720.

Cos π *u*.......................................Sheppard, 1899

"On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation," *Phliosophical Transactions*, A, CXCII, 141.

Critical ratio....................................McGaughy, 1924

Gauss, Quetelet, Galton, and others used the fraction $\frac{\text{deviation}}{\text{probable error of deviation}}$ as an argument from which to determine the probability of the occurrence of an error of a particular magnitude. McGaughy (*Fiscal Administration of City School Systems*) proposes to make a sharp dichotomy, reporting merely that this fraction is or is not greater than 3.

Curve of rank relation....................................Otis, 1916

"Some Logical Aspects of the Binet Scale," *Psychological Review*, XXIII, 720.

Curve of relation....................................Otis, 1916

"Some Logical Aspects of the Binet Scale," *Psychological Review*, XXIII, 720. Thorndike used the term *central relation line* and *relation line* in *Mental and Social Measurements*, p. 151.

D (= *the 10–90 percentile range*)............................Kelley, 1921

"A New Measure of Dispersion," *Publications of the American Statistical Association*, XVII (1921), 743–749.

† *Decile*..Galton

The term occurs in "Some Results of the Anthropometric Laboratory," *Journal of the Anthropological Institute*, XIV (1885), 276–288, and perhaps earlier. See Chapter IV on "Percentiles."

Difference formula.

To one of the difference methods for computing the product-moment coefficient of correlation, published by Pearson in 1907, and called by him the *variate difference method*, (q. v.), Otis applied the term *deviation formula* and later, *difference formula*. The formula does not, as Otis erroneously implies, furnish a new method of measuring correlation, but merely a different technique of computing the product moment coefficient. See Otis, "The Reliability of the Binet Scale and of Pedagogical Scales," *Journal of Educational Research*, IV (1921), (an article written in 1916) and also *Statistical Method in Educational Measurement* (1925). See also page 129 of this present work.

Difference methods (*of computing the Pearson r*).............Pearson, 1907

"On Further Methods of Determining Correlation," *Drapers' Company Research Memoirs, Biometric Series*, IV, p. 4.

† *Dispersion.*

The term is used by Lexis in *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft,* (1877), and may have been used by others earlier.

* *Educational quotient*. . . . . Franzen, 1920

"The Accomplishment Quotient" (*Teachers College Record,* 1920) contains this term, the abbreviation EQ, and many other similar abbreviations such as AQ, VQ, and SQ. See discussion of IQ.

Eta (*for correlation in a non-linear system*). . . . . Pearson, 1903

In a footnote on p. 304 of "Mathematical Contributions to the Theory of Evolution.—On Homotyposis in Homologous but Differentiated Organs," *Proceedings of the Royal Society,* LXXI (1903), he defines η as $\frac{\sigma_M}{\sigma}$, and discusses briefly its meaning. He does not use the term *correlation ratio* here.

Experimental coefficient. . . . . McCall, 1923

Beginning with De Moivre (1738), who divided his deviations by σ, and Kramp (1799), whose divisor was $\sigma\sqrt{2}$, workers in probability have all made use of some such ratio. McCall's use of the *experimental coefficient* is similar to McGaughy's use of the *critical ratio,* the only difference being that the denominator of the EC is not the probable error, but 2.78 σ.

Footrule. . . . . Spearman, 1906

The formula $R = 1 - \frac{3\,\Sigma\, d}{n^2 - 1}$ was published by Spearman in "The Proof and Measurement of Association between Two Things," *American Journal of Psychology,* XV (1904), 87, where it is called the *method of rank differences.* The term is found in " 'Footrule' for Measuring Correlation," *British Journal of Psychology,* II (1906), 89–108.

Geometric mean.

See *mean.*

Grade-variate correlation. . . . . Pearson, 1907

$$r_{12} = 2\, sin\left(\frac{\pi}{6}\,\rho_{12}\right)$$

"Mathematical Contributions to the Theory of Evolution.—XVI. On further Methods of Determining Correlation," *Drapers' Company Research Memoirs, Biometric Series,* IV (1907), 12.

Harmonic mean.

See *mean.*

Heteroclitic. . . . . Pearson, 1905

See *homoclitic.*

Heteroscedastic. . . . . Pearson, 1905

See *homoscedastic.*

Heteroclisy. . . . . Pearson, 1905

"Mathematical Contributions to the Theory of Evolution.—XIV. On the General

Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 23.

κλισις = inclination.

See *homoclitic*.

Hierarchical order of correlation coefficients...............Spearman, 1904

"General Intelligence Objectively Determined and Measured," *American Journal of Psychology*, XV (1904).

Homoclisy..Pearson, 1905

See *heteroclisy*.

Homoclitic..Pearson, 1905

"Lastly, we want a word to express the idea of all the arrays having equal skewness, or being asymmetrical in an equal degree about their means. I shall express this by the term *homoclitic*; generally the arrays will not be equally asymmetrical round their means, and in this case we shall speak of them as *heteroclitic*." "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 22, 23.

† *Homograde and heterograde statistics*..........................Charlier

The terms occur in *Vorlesungen über die Grundzüge der Mathematischen Statistik*, (2d ed, 1920), and perhaps earlier.

Homoscedastic..Pearson, 1905

"If . . . . all arrays are *equally scattered* about their means, I shall speak of the system as a *homoscedastic* system, otherwise it is a *heteroscedastic* system." "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 22.

Homoscedasticity..Pearson, 1905

"Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 23.

Index of correlation..Galton, 1888

"Co-relations and their Measurements, chiefly from Anthropological Data," *Proceedings of the Royal Society*, XLV (1888–89), 143.

Index of reliability.

The relationship $r_{pq} = \sqrt{r_1}$, where r_{pq} is the probable correlation of any one test with the average of an indefinitely large number of similar tests, and r_1 is the reliability coefficient of the test in question, was published by Abelson in 1911 ("Mental Ability of Backward Children," *British Journal of Psychology*, IV, 392). It was derived from a longer formula of Spearman's and may have been due to him. It was again developed independently by Kelley in "A Simplified Method of Using Scaled Data for Purposes of Testing," *School and Society*, IV (1916). The term *index of reliability* was applied to it by Monroe in *An Introduction to the Theory of Educational Measurement* (1923), and seems to have been suggested by a phrase in Kelley's paper.

IQ. . . . . .Stern and Bobertag

Bobertag is probably the originator of the type of abbreviation employed in such expressions as MA, CA, IQ, EQ, EA, and the like, a type of abbreviation which is clear when it occurs in a paragraph of prose, but which is confusing when placed in a mathematical formula. In 1911, in the first section of his paper, "Über Intelligenzprüfung," in the *Zeitschrift für angewandte Psychologie*, V (1911), Bobertag set down the following scheme of abbreviations:

IP. = Intelligenzprüfung	Kn. = Knaben
Vp. = Versuchsperson	Md. = Mädchen
Kd = Kinder	B.-S. = Binet und Simon.

The following year, Stern proposed the idea of the intelligence quotient, the ratio of mental to chronological age, and used the terms *Lebensalter*, *Intelligenzalter*, *Alterstufe*, *Intelligenzstufe*, and *Intelligenzquotient*, but not the abbreviations for them, in his book *Psychologischen Methoden der Intelligenzprüfung* (Leipzig, 1912). Almost immediately Bobertag made use of the new concept, acknowledging Stern's suggestion. In a footnote on the first page of the second part of his paper Über Intelligenzprufung" in the *Zeitschrift*, VI (1912), 405–538, he says: "Ich verwende folgende Abkürzungen:

LA. = Lebensalter	IS. = Intelligenzstufe
IA. = Intelligenzalter	IQ. = Intelligenzquotient
AS. = Alterstufe	FK. = (Gaussische) Fehlerkurve."

Thus our CA corresponds to his LA., our MA to his IA., and our IQ to his IQ.

The terms do not seem to have been used in English before 1914. In that year, in a paper called "One Hundred Juvenile Delinquents Tested by the Binet Scale," (*Pedagogical Seminary*, XXI, 523–531) Pintner wrote, "The Intelligence Quotient, which was also calculated for all cases, has been advocated by Stern and employed by Bobertag." In the same year Whipple made an English translation of Stern's work, *Psychological Methods of Testing Intelligence* (Baltimore, 1914) and used the expressions *mental age, mental retardation, and mental quotient.*

Isokurtic. . . . . .Pearson, 1905

"I term arrays of no skewness *isokurtic*, and skew arrays *allokurtic*." "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 23.

Kurtosis. . . . . .Pearson

"Given two frequency distributions which have the same variability as measured by the standard deviation, they may be relatively more or less flat-topped than the normal curve. If more flat-topped I term them *platykurtic*, if less flat-topped *leptokurtic*, and if equally flat-topped *mesokurtic*. A frequency distribution may be symmetrical, satisfying both the first two conditions for normality, but it may fail to be *mesokurtic*, and thus the Gaussian curve cannot describe it." "Skew Variation, A Rejoinder," *Biometrika*, IV (1906), 173. The measure of kurtosis is given as β_2-3.

κυρτότης = a curving or arching, or in mathematics, convexity.
λεπτό = fine, thin, small.
πλατύ = broad, flat.
κυρτός = convex.

Leptokurtic. . . . . .Pearson

See *kurtosis*.

Lexian ratio.

The ratio was proposed by Lexis in 1877, in his book, *Zur Theorie der Massenerscheinungen*, p. 28.

Lexian series.

See *Poisson series*.

MA.

See *CA*.

Mass measures. . . . . .McCall

"What statistical measure to compute, whether to compute any measure at all, how to interpret the statistical measures when computed, all three questions depend for their answer in part upon one of the following three mass measures, especially the first or second.

I. Frequency Surface.

II. Frequency Distribution.

III. Order Distribution."

How to Measure in Education (1923), p. 354.

Mean.

This is a very old term. Heath, in his *History of Greek Mathematics* (Oxford, I, 1921, 85) says: "We are told that in Pythagoras's time there were three means, the arithmetic, the geometric and the subcontrary, and that the name of the third (subcontrary) was changed by Archytas and Hippasus to 'harmonic.' A fragment of Archytas's work *On Music* actually defines the three; we have the arithmetic mean when, of three terms, the first exceeds the second by the same amount as the second exceeds the third; and the geometric mean when, of three terms, the first is to the second as the second is to the third; the '*subcontrary* which we call the *harmonic*,' when the three terms are such that 'by whatever part of itself the first exceeds the second, the second exceeds the third by the same part of the third.' " Various other means have been used besides these three familiar forms, and the number of possible definitions of a mean is probably inexhaustible.

Mean contingency. . . . . .Pearson, 1904

"Mathematical Contributions to the Theory of Evolution.—XIII. On Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series*, I (1904), 7.

Mean deviation.

See *mean error*.

Mean error. . . . . .Pearson, 1893

Pearson used the term *mean error* in the sense of the mathematical mean of the errors, theoretically equal to .7979 σ. Before this time the term had been used for the value which we now call the standard deviation, and which the Germans called *der mittlere Abweichung*. See "Contributions to the Mathematical Theory of Evolution.—I. On the Dissection of Asymmetrical Frequency Curves," *Philosophical Transactions*, A, CLXXXV (1894), Part 1, p. 102. The paper was read in 1893.

Mean square contingency..Pearson, 1904

"Mathematical Contributions to the Theory of Evolution.—XIII. On Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series*, I (1904), 6.

Mean square contingency coefficient........................Pearson, 1904

"Mathematical Contributions to the Theory of Evolution.—XIII. On Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series*, I (1904), 16.

† *Median*..Galton

"In consequence of the multitude of mediocre values, we always find that on either side of the middlemost ordinate . . . . which is the median value and may be accepted as the average, there is a much less rapid change of weight than elsewhere." *Inquiries into Human Faculty*, Chapter on "Statistical Methods," 1883. (In 2d. ed., p. 35.) Galton had employed the concept of the median as early as 1869, but not the name. Fechner called this measure *der Centralwerth* or C, and gave a complete treatment of its properties and computation in his memoir "Ueber den Ausgangswerth der kleinsten Abweichungssumme," *Abhandlungen der Königlichen Sächsichen Gesellschaft der Wissenschaften*, Bd. II, Abt. I, 1874.

* *Median ratio coefficient*................................Thorndike, 1907

Empirical Studies in the Theory of Measurement. This is very similar to Galton's earliest method of calculating correlation.

Mentaces..Pearson, 1907

"I divide the range of the 'Intelligent' into a hundred units, which I propose to term *mentaces*. . . . . Clearly 100 mentaces is not far from the standard deviation of intellectual power in man." "On the Relationship of Intelligence to Size and Shape of Head and to other Physical and Mental Characters," *Biometrika*, V, 110–111.

Mesokurtic..Pearson

See *kurtosis*.

Mode..Pearson, 1894

"I have found it convenient to use the term *mode* for the abscissa corresponding to the ordinate of maximum frequency." "Contributions to the Mathematical Theory of Evolution.—II. Skew Variation in Homogeneous Material," *Philosophical Transactions*, A, CLXXXVI (1895), Part 1, 345. Fechner applied the term *der dichteste Werth* to the same magnitude in 1878.

Modulus...De Moivre, 1838

The value of the modulus here is 2 σ, *Doctrine of Chances*, 2d ed. (1738), 234–243.

* *Modulus*..Airy

Theory of Errors of Observation, 2d ed. revised (1875). Here *modulus* has the usual meaning of $\sigma\sqrt{2}$.

Moment.

The concept has been utilized by various writers, some of whom have also used the word *moment* and have definitely called attention to the analogy to mechanics. Among these are Quetelet and De Forest. The mathematical values of the first six moments of the normal curve were published by Kramp in 1799, and were used

by Gauss, Encke, and Czuber. Pearson's first use of the term to indicate a definite statistical concept seems to have been in a letter to *Nature* in 1893. Here he used μ to represent a moment around the mean.

† *Moment coefficient*..Pearson

This occurs in the editorial "On the Probable Errors of Frequency Constants, II," *Biometrika*, IX (1913), 1–10, an article reproduced from lecture notes. The term had probably been in use some time previously.

Multiple contingency..................................Pearson, 1904

"Mathematical Contributions to the Theory of Evolution.—XIII. On Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series*, I (1904), 22.

Nomic..Pearson, 1905

"A heteroclitic system of arrays may be *nomic* or *anomic*, according as the skewness of the arrays changes continuously or irregularly with the position of the array. A heteroscedastic system of arrays is also either *nomic* or *anomic*, according as the standard deviation of the arrays changes continuously or irregularly with the position of the arrays." "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 51.

Normal curve.

The formula was discovered by De Moivre in 1733. It has been called by many different names, such as *curve of error*, *curve of facility of error*, *das Fehlerkurve*, *Gaussian curve*, *Laplace-Gaussian curve*, *probability curve*, and many others. The origin of the use of the word "normal" to describe the curve is obscure. Galton used it, as did also Lexis, and the writer has not found any reference which seems to be its first use. It is not improbable that the term goes back to Quetelet.

Octile.

See *quartile*.

Ogive curve..Galton

The term was borrowed by Galton from architecture. "Statistics by Intercomparison, with Remarks on the Law of Frequency of Error," *Philosophical Magazine*, 4th series, XLIX (1875), 33 *et seq.*

Parabola of regression..................................Pearson, 1905

"Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 30.

Partial correlation......................................Pearson, 1897

"These values have been termed by Mr. G. U. Yule *nett* coefficients of correlation, to distinguish them from r_{23}, r_{13}, and r_{12}, which he terms *gross* coefficients. The difference would, perhaps, be best expressed mathematically by the use of such terms as *partial* correlation coefficient and *total* correlation coefficient, the former being the value of the coefficient when one variable is not allowed to vary, and the latter when it is." Karl Pearson and Alice Lee: "On the Distribution of Frequency (Variation and Correlation) of the Barometric Heights at Divers Stations," *Philosophical Transactions*, A, CXC (1897), 462. For the history of the concept of partial correlation,

see p. 111. Galton had used the term "partial co-relation" in 1888, but not exactly in the present sense.

† *Percentile*. . . . . Galton

"The value that is un-reached by n per cent of any large group of measurements, and surpassed by 100-n of them, is called its nth per-centile." This definition occurs in the heading for a table in "Some Results of the Anthropometric Laboratory," *Journal of the Anthropological Institute*, XIV (1885), 277.

Platykurtic. . . . . Pearson

See *kurtosis*.

Point measures. . . . . McCall

"Mass measures may be vague and statistically cumbersome, but they possess the virtue of including every score in the class. It is the function of point measures to represent the condition of a class by a single number. The point chosen to represent the class depends upon the statistical method employed. The common methods are: I. Mode; II. Mean; III. Median or Midscore; IV. Lower Quartile Point; V. Upper Quartile Point." *How to Measure in Education* (1923), p. 365.

The term *point measure* has not come into general use. This is probably because, strictly speaking, a point has no size and cannot be considered as a measure, and the geometric representation of such measures as those named above is more properly a line than a point.

Poisson series.

The differentiation between the three types of statistical series, Bernoullian, Lexian, and Poisson, was made by Lexis in 1877 in his book, *Zur Theorie der Massenerscheinungen*. The terminology seems to be due to Charlier.

Predictive index. . . . . Bailor, 1924

Content and Form in Tests of Intelligence, Teachers College, Columbia University, Contributions to Education, No. 162.

Probable error. . . . . due to Bessel, 1815

"Ueber den Ort des Polarsterns," *Berliner Astronomishes Jahrbuch für 1818*. Bessel here introduced the term *der wahrscheinliche Fehler*. Gauss used the same expression the next year in "Bestimmung der Genauigkeit der Beobachtungen" (*Zeitschrift für Astronomie und verwandte Wissenschaften*, I), where he has $r = 0.6744897\sqrt{\frac{\Sigma x^2}{n}}$. When writing in Latin after this date, Gauss used *error probabilis*. The French sometimes use *l'erreur probabile*.

† *Product-moment*. . . . . Pearson

The term was used by Pearson in 1905 ("On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs, Biometric Series*, II) and had probably been employed by him before that. In this paper he uses the p notation as though already well known.

† Q (*for semi-interquartile range*). . . . . Galton

The symbol is used in *Natural Inheritance* (1889), and probably earlier.

Q (*for coefficient of association*). . . . . Yule, 1900

See *coefficient of association*.

Quartile.

McAlister in his paper on "The Law of the Geometric Mean" (*Proceedings of the Royal Society*, XXIX, 1879) has the following passage on page 374: "Among the measures greater than the mean there is one which may be called middlemost: *i.e.* such that it is an even wager that a measure (greater than the mean) lies above it or below it. A similar middlemost measure exists among those that are less than the mean. As these two measures, with the mean, divide the curve of facility into four equal parts, I propose to call them the '*higher quartile*' and the '*lower quartile*' respectively. It will be seen that they correspond to the ill-named '*probable errors*' of the ordinary theory. . . . . Similarly between zero and lower quartile we place a midmeasure which we call the '*lower octile.*' The '*higher octile*' will subdivide the interval between the higher quartile and infinity." McAlister used the symbols Q and q for the two quartiles. This paper follows a brief paper by Galton on the same subject, who writes ". . . . and subsequently I will communicate a memoir by Mr. Donald McAlister, who, at my suggestion, has mathematically investigated the subject." Whether Galton or McAlister coined the term "quartile" the writer does not know. Certainly it was Galton who brought it into general use.

r (for coefficient of correlation).................................Galton, 1877

The letter was used by Galton in a lecture on "Typical Laws of Heredity in Man" before the Royal Institution, Feb. 9, 1877. It was used here for *reversion* which he later changed to *regression*. In 1886 he used w, writing $f = p\sqrt{1 - w^2}$, on page 60 of "Family Likeness in Stature." In 1888 he returned to the original r, which may be found in the heading of one column in Table V, page 143, of "Co-relations and their Measurement," *Proceedings of the Royal Society*, XLV, 135–145. Edgeworth used ρ in his paper on "Correlated Averages," 1892.

Regression..Galton, 1877

Address on "Typical Laws of Heredity in Man," made before the Royal Institution, Feb. 9, 1877—unpublished. The term used here was *reversion*. See his Presidential address on law of regression made before Section H of the British Association, at Aberdeen, 1885, printed in *Nature*, Sept. 1885. See also "Regression towards Mediocrity in Hereditary Stature," *Journal of the Anthropological Institute*, XV (1885), 246–264.

Reliability coefficient.................................Spearman 1910

Spearman used the term to describe the correlation between two comparable measures of the same trait, and indicated the coefficient by the symbol $r_{x_1x_2}$. He had employed the concept but not the term in 1904 and 1907. In 1911 Brown used the same term to describe the correlation between two repetitions of the same test form, but later returned to Spearman's original definition, which is the interpretation of the term current today. The first use of the term is in "Correlation Calculated from Faulty Data," *British Journal of Psychology*, III.

Scedastic curve..Pearson, 1905

"The curve in which the ratio of the standard deviation of the array to the standard deviation of the character in the population at large is plotted to position is termed a *scedastic curve*" (that is, a curve which measures the "scatter" in the arrays). "Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905), 52.

Scedasticity. . . . . Pearson, 1905

"Mathematical Contributions to the Theory of Evolution.—XIV. On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, II (1905). σκέδασις = a scattering. σκεδαστὸς = capable of being scattered. Compare "skedaddle," which the *Oxford Dictionary* says is of uncertain origin.

* *Self-correlation*. . . . . McCall

McCall uses the term to mean either a reliability coefficient or a retesting coefficient, no distinction being made between the two. On page 310 in *How to Measure in Education*, he says: "The method of self-correlation is to compute the coefficient of correlation between the two series of scores secured from two administrations of the same or duplicate tests to the same pupils."

Sigma (for the standard deviation). . . . . Pearson, 1893

See *standard deviation*

Square contingency. . . . . Pearson, 1904

"Mathematical Contributions to the Theory of Evolution.—XIII. On the Theory of Contingency and its Relation to Association and Normal Correlation," *Drapers' Company Research Memoirs, Biometric Series*, I, 15.

Standard deviation. . . . . Pearson, 1893

"Contributions to the Mathematical Theory of Evolution.—I. On the Dissection of Asymmetrical Frequency Curves," *Philosophical Transactions*, A, CLXXXV (1894), Part I, 80. The symbol σ is used here for the first time in this connection.

Standard error. . . . . Yule, 1897

In 1877 Galton had recognised the relationship $v = c\sqrt{1 - r^2}$. Yule applied the term in his paper "On the Theory of Correlation," *Journal of the Royal Statistical Society*, LX, 821–851.

Standard measure.

Almost all writers on probability from the time of De Moivre have expressed their variables as multiples of some intrinsic unit. Gauss divided his deviations by the modulus, $\sigma\sqrt{2}$, and Galton by the probable error, or by the quartile. Since the publication of Sheppard's *Tables of the Probability Integral* (1903), the standard deviation has been used in preference to either of these units. Kelley has termed the ratio $\frac{x}{\sigma}$ a *standard measure*. "Comparable Measures," *Journal of Educational Psychology* (1914), 589–595.

Standard normal curve. . . . . Sheppard, 1899

The term is used for a normal curve whose area and standard deviation are unity, the curve which forms the basis for the tables of the probability integral computed by Sheppard and now in common use. "On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation," *Philosophical Transactions*, A, CXCII (1899), 105–106.

† *Statistic*. . . . . R. A. Fisher

"A *statistic* is a value calculated from an observed sample with a view to characterising the population from which it is drawn. . . . . The utility of any particular statistic, and the nature of its distribution, both depend on the original distribution. . . . ." *Statistical Methods for Research Workers* (1925), p. 44.

Tetrachoric functions. . . . . P. F. Everitt, 1910

The term appears to have been used for the first time in an article, "Tables of the Tetrachoric Functions for Fourfold Correlation Tables," *Biometrika*, VII, 437–441. The formula (but not the name) was given by Pearson in "Mathematical Contributions to the Theory of Evolution.—VII. On the Correlation of Characters not Quantitatively Measurable," *Philosophical Transactions*, A, CXCV (1901), 1–47.

Total correlation. . . . . Pearson, 1897

See *partial correlation*.

T-Scale. . . . . McCall, 1923

How to Measure in Education, Chapter X.

Two factors.

The theory is due to Spearman, but the name was furnished by Sancti de Sanctis, 1913.

Type. . . . . Pearson, 1895

"Mathematical Contributions to the Theory of Evolution.—III. Regression, Heredity, and Panmixia," *Philosophical Transactions*, A, CLXXXVII (1896), 253–318. Pearson uses *type* for that value of one variate with which an array of the other is associated.

Variance. . . . . R. A. Fisher, 1918

"It is therefore desirable in analysing the causes of variability to deal with the square of the standard deviation as the measure of variability. We shall term this quantity the Variance." "The Correlation between Relations on the Supposition of Mendelian Inheritance," *Transactions of the Royal Society of Edinburgh*, LII, 399.

Variate difference method (of computing correlation).

Pearson gave the formula $r_{xy} = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_v^2}{2\sigma_x\sigma_y}$ where $v = x - y$, in 1907, in his paper "On Further Methods of Determining Correlation," *Drapers' Company Research Memoirs, Biometric Series*, IV, 4. He stated here that "this method of finding r_{xy} has long been in use as an alternative method to the product-moment method. . . ." Pearson had published a formula very similar to this one in 1896 and in 1902 (*Philosophical Transactions*, CXCVIII, 242. See page 129.

APPENDIX

The Study of College Catalogues

The schools originally chosen for the catalogue study were: University of Chicago, Clark, Columbia, Harvard, University of Illinois, University of Iowa, University of Michigan, University of Minnesota, University of Pennsylvania, Stanford, Yale. Stanford and Minnesota were later excluded from consideration because in the earlier years their courses in statistics were relatively unstable, shifting about from one instructor to another, or disappearing altogether for brief intervals. To have covered all the institutions where statistics was taught at an early date, it now appears that the selection should have included several other universities, and some of the technological institutes.

A list was made year by year and school by school, of all courses which appeared to have statistical content, with the name of the instructor, the department in which the course was given, and any other essential facts which the catalogue description might reveal.

A large chart was then made, showing the distribution of these courses by school and by year. In each cell of the table thus formed were recorded the names of the men who were teaching some form of statistics in that school and that year. The use of different colored pencils to indicate different departments made it possible to introduce a third dimension into the chart.

A chart was then made for each college showing the yearly offerings of courses in each of the six major groups of courses. Finally a chart was made for each major subject showing the schools which presented courses in statistics from the point of view of that department for each year in the period.

These charts have not been reproduced here because they are believed to contain inaccuracies which could not be removed without an expenditure of labor which the importance of the inquiry does not appear to warrant. The reasons for these inaccuracies are:

1. The catalogue titles of courses are often misleading. It is continually necessary to go back of the printed title to a knowledge of the writings of the instructor in question. Sometimes the title of a course—"Child Study" or "Physical Anthropology"—has been recorded on the charts because of personal information about the course or because a text written

by the instructor at that time and on a topic closely related to the subject of the course placed an emphasis on the use of statistical methods. Again another course with the same name, but taught by a man who had never himself made use of statistical methods, has been omitted from the records. The task was made somewhat less difficult by the fact that until about 1915 the list of course offerings of even our largest schools was short enough to permit the printing of a fairly full description of the courses.

2. Catalogue offerings for any given year cannot be relied upon to be identical with courses actually taught in that year. When the catalogue prints the name of a course without the name of any instructor, there is at least a reasonable basis for doubt as to whether the course was given. Furthermore, names are occasionally retained a year or two after the course is abandoned.

3. It is not always possible to be sure whether announcements relate to the current or the coming year. This might cause an error of a year in date, but could not vitiate general conclusions.

4. To compile a list of courses labeled in the catalogues as "Educational Statistics" would be a relatively easy task, but the results would be trivial if not actually misleading, inasmuch as some well-known educators caught their first suggestion of the value of statistical methods of research from someone who was teaching anthropology, psychology, zoölogy, or mathematics. Therefore any course has been recorded if the catalogue description indicates that it presented methods of dealing quantitatively with the variable results of observations, no matter whether the course was characterized as "Child Study," "Thremmatology," "Calculus of Observations," "Psycho-Physical Methods," or was described in some other way.

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The bibliography presented here differs considerably in its emphasis from bibliographies on statistics which have been recently published by others. It also differs from the bibliography which the present writer would construct if the importance of a work to the theory of statistics were the sole criterion for selection. Any book or paper whose title appears in this list has been actually consulted during the preparation of the study. The converse is, however, not true, for numerous works were found to contain so little which was pertinent to this inquiry that it seemed wise not to include them in the list. It is apparent that many important books and papers have been omitted. Inasmuch as other bibliographies are available, it seemed wise not to add to this one any titles—however important—which had not been consulted by the writer, and to assume that a complete bibliography would be of less service to the reader than a bibliography which would permit him to evaluate the background of reading from which the study was prepared.

No attempt has been made to enumerate the sources of biographical material. For the most part these were the standard encyclopaedias, English, German, and French, *Who's Who*, *Wer Ist's*, and similar works, and Smith's *History of Mathematics*, Vol. I.

The symbolism IV (1899), 563–674 has been generally employed for Volume IV, pages 563 to 674, of a periodical published in 1899. In a few cases when the magazine is not generally well known, the place of publication has been added, also.

The journal here referred to as the *Philosophical Magazine* is the *London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, a publication which has run through numerous series with slight changes in title. The full title of the journal referred to as *Philosophical Transactions* is the *Philosophical Transactions of the Royal Society of London*. The publication now known as the *Journal of the American Statistical Association* has had various titles, as follows:

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Sans Tache

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IN THE "elder days of art" each artist or craftsman enjoyed the privilege of independent creation. He carried through a process of manufacture from beginning to end. The scribe of the days before the printing press was such a craftsman. So was the printer in the days before the machine process. He stood or fell, as a craftsman, by the merit or demerit of his finished product.

Modern machine production has added much to the worker's productivity and to his material welfare; but it has deprived him of the old creative distinctiveness. His work is merged in the work of the team, and lost sight of as something representing him and his personality.

Many hands and minds contribute to the manufacture of a book, in this day of specialization. There are seven distinct major processes in the making of a book: The type must first be set; by the monotype method, there are two processes, the "keyboarding" of the MS and the casting of the type from the perforated paper rolls thus produced. Formulas and other intricate work must be hand-set; then the whole brought together ("composed") in its true order, made into pages and forms. The results must be checked by proof reading at each stage. Then comes the "make-ready" and press-run and finally the binding into volumes.

All of these processes, except that of binding into cloth or leather covers, are carried on under our roof.

The motto of the Waverly Press is *Sans Tache.* Our ideal is to manufacture books *"without blemish"*—worthy books, worthily printed, with worthy typography—books to which we shall be proud to attach our imprint, made by craftsmen who are willing to accept open responsibility for their work, and who are entitled to credit for creditable performance.

The printing craftsman of today is quite as much a craftsman as his predecessor. There is quite as much discrimination between poor work and good. We are of the opinion that the individuality of the worker should not be wholly lost. The members of our staff who have contributed their skill of hand and brain to this volume are:

Keyboard: Mary Frank, Hannah Scott, Katharine Kocent, Louise Hilpert, Helen Twardowicz.

Casters: Ernest Wann, Charles Aher, Kenneth Brown, George Bullinger, Norwood Eaton, Charles Fick, Martin Griffen, Henry Lee, Mahlon Robinson, George Smith.

Composing Room: Harry Harmeyer, William Sanders, Arthur Baker, Edward Rice, Anthony Wagner, Ray Kauffman, Andrew Rassa, Herbert Leitch, Charles Wyatt, Henry Johansen, Theodore Nilson, James Jackson, John Crabill, Richard King, George Moss, Henry Shea.

Proof Room: Sarah Katzin, Alice Reuter, Mary Reed, Ruth Jones, Ethel Strasinger, Lucile Bull, Angeline Eifert, Audrey Tanner, Dorothy Strasinger, Lillian Gilland, Shirley Seidel, Mary Williams, Mary Stanton.

Cutter: William Armiger.

Press: Robert Gallagher, Raymond Gallagher, Henry Augsburg, Hugh Gardner, Thomas Shreck, Andrew Becker, Wm. Harrison, Jr.

Folder: Laurence Krug, Marvin Witt.